Modeling Rarefaction and Shock waves

Dallas Gosselin and Jonathan Fernandez

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Introduction

Rarefaction and shock waves are combinations of two wave fronts created from the initial disturbance of the medium. Each of the wave front travels at a different speed for $t \geq 0$. In rarefaction waves the leading wave travels faster than the trailing wave, causing decompression of the medium. For shock waves the leading wave front travels slower, therefore causing compression of the medium.

Both waves are modeled by nonlinear hyperbolic PDEs which can be solved through the method of characteristics. The simplest PDE model is given by

$$u_t + a(u)u_x = 0.$$
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The simplest PDE model is given by $u_t + a(u)u_x = 0$. 
Solving the PDE!

Let's first assume $a$ is a function of $x$ and $t$. Now the characteristic curve solves the ODE $\frac{dx}{dt} = a(x, t)$. Every characteristic curve is unique passing through some $(x_0, t_0)$ and along each curve

$$0 = u_x a(x, t) + u_t = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} = \frac{du}{dt}$$

This implies $u$ is constant along every curve.

Example: Let $a(x, t) = e^x + t$, then $\frac{dx}{dt} = e^x + t$. The solution to this are given by $e^{-x} = -e^t + C$. Since $u$ is constant along each curve, we can write the solution as $u(x, t) = f(C) = f(e^{-x} + e^t)$. 

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$$u(x, t) = f(C) = f(e^{-x} + e^t).$$
Now suppose \( a \) is a function of \( u \), such that \( u \) is constant along every characteristic curve. In particular let's take the inviscid Burger's equation:

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    u_t + uu_x = 0.
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In this case there are three principles that hold:

1. Each characteristic curve is a straight line.
2. The solution is constant on each such line.
3. The slope of each such line is equal to the value of $u$ on it.

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(c) *The slope of each such line is equal to the value of \( u(x, t) \) on it.*

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The Problem

Let's impose the initial condition that \( u(x,0) = \phi(x) \). By the third principle mentioned in the last slide, the slope of any characteristic line going through \((x_0,0)\) will equal \( \phi(x_0) \). Problem arises when characteristic lines intersect, which is an issue because we cannot have multiple slopes at the same point.
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Avoiding the Problem

We can prevent any intersection of any characteristic by assuming $\phi(x)$ never decreases for $t \geq 0$, hence the slope of the characteristic is never decreasing. This causes rarefaction waves.

We can allow discontinuities in the solution that occur in the solution. This creates a shock wave.

We assume the solution exists near $t = 0$ and breaks down away from $t = 0$. 

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Rarefaction Waves

Let's assume $\phi(x)$ never decreases. Since the characteristic lines do not intersect, we can write the slope of any characteristic line going through $(z, 0)$ and $(x, t)$ as

$$\frac{x-z}{t-0} = \frac{dx}{dt} = a(u(x, t)) = a(u(z, 0)) = a(\phi(z))$$
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This implies that $u(x, t) = \phi(z(x, t))$, where $x - z = ta(\phi(z))$, with the condition $a(\phi(z)) \leq a(\phi(w))$ for $z \leq w$. 
Grigoryan’s Example!

Take the initial condition,

\[ \phi(x) = \begin{cases} 
1 & \text{if } x < 0 \\
2 & \text{if } x > 0
\end{cases} \]  

(0.1)

The slopes of the characteristic lines are given by \( \frac{dx}{dt} = \phi(x(t)) = \phi(x(0)) \).

The waves that originate from \( x(0) < 0 \) have greater speed than those originating from \( x(0) > 0 \), creating an increasing gap.
The solution to a shock wave system, referred to as the observed jump discontinuity, acquires multiple values corresponding to each line at the crossing point. Herein lies the dilemma of shock waves, where the mathematical model we have developed breaks down and the solutions are no longer uniquely determined.
Grigoryan’s Example Revisited!

Take the new initial condition for the inviscid Burger’s equation,

\[ \phi(x) = \begin{cases} 
  2 & \text{if } x < 0 \\
  1 & \text{if } x > 0 
\end{cases} \]  

(0.2)

The slopes of the characteristic lines are once again given by

\[ \frac{dx}{dt} = \phi(x(t)) = \phi(x(0)). \]

The waves here, however, that originate from \( x(0) < 0 \) have a slower speed than those originating from \( x(0) > 0 \), creating somewhat of a wedge where the characteristic lines intersect.
The jump discontinuity we observe at $x = 0$ implies that the lines have a smaller slope for $x(0) < 0$ and greater slopes for $x(0) > 0$, thereby revisiting our previous claim that shock waves take the form of the leading wave front traveling slower than the trailing wave front.

The decreasing initial condition then provides us with a region in the plane where the intersection of the characteristic lines with different slopes, produce multiple solutions.
Let $a(u) = A'(u)$ in our model PDE, thereby allowing the equation to be rewritten as:

$$u_t + A(u)_x = 0.$$  \hspace{1cm} (0.3)

We now require the above equation to hold in the sense of distributions, such that

$$\int_0^\infty \int_{-\infty}^\infty [u\psi_t + A(u)\psi_x] \, dx \, dt = 0$$

for all test functions $\psi(x, t)$. 

Now we make the conjecture that the shock occurs along the curve $x = \xi(t)$ in the figure below.

Notice Because the shock occurs at the jump discontinuity of our step function, the left and right limits are defined as $u^-(t) = u(x^-, t)$ and $u^+(t) = u(x^+, t)$ respectively.
By taking equation
\[ \int_0^\infty \int_{-\infty}^{\infty} \left[ u \psi_t + A(u) \psi_x \right] \, dx \, dt = 0 \]
and splitting the inner integral into two separate definite integrals followed by applying the derivative of distribution we obtain the following
\[
\int_0^\infty \int_{-\infty}^{\xi(t)} \left[ -u_t \psi - A(u)_x \psi \right] \, dx \, dt + \int_0^\infty \int_{\xi(t)}^{\infty} \left[ -u_t \psi - A(u)_x \psi \right] \, dx \, dt = 0.
\]

- The divergence of some vector field \( \vec{F} = (-u \psi, -A(u) \psi) \).

- By the divergence theorem, we can rewrite each double integral as a single integral over the line \( x = \xi(t) \):
\[
\int_{x=\xi(t)} \left[ u^+ \psi n_t + A(u^+) \psi n_x \right] \, dl = \int_{x=\xi(t)} \left[ u^- \psi n_t + A(u^-) \psi n_x \right] \, dl = 0
\]
By setting the integrands of
\[
\int_{x=\xi(t)} \left[ u^+ \psi n_t + A(u^+) \psi n_x \right] \, dl = \int_{x=\xi(t)} \left[ u^- \psi n_t + A(u^-) \psi n_x \right] \, dl = 0
\]
equal to one another and applying some minor algebra we get the following result

\[
\frac{A(u^+) - A(u^-)}{u^+ - u^-} = -\frac{n_t}{n_x} = s(t).
\]

Where the equation above is defined as the Rankine-Hugoniot formula for the speed of the shock wave, \( s(t) \).
We have shown that rarefaction and shock waves are combinations of two wave fronts created from an initial disturbance of the medium. In rarefaction waves the leading wave travels faster than the trailing wave, while for shock waves the leading wave front travels slower. For a shock wave to exist, the characteristic lines must intersect and the characteristic behind the shock must have a greater slope than the characteristic after the shock. Mathematically this is represented by the a jump discontinuity that satisfies both Rankine-Hugoniot’s formula and the following result:

$$a(u + s) < s(t) < a(u - s)$$
## Conclusion

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\[ a(u^+) < s(t) < a(u^-) \]
PDEs!!
For our sources please refer to our paper.
-Thanks