Modeling Rarefaction and Shock Waves

In our universe, there are many types of natural phenomena which can be modeled by nonlinear partial differential equations (PDE). These PDEs can be rather difficult to model and even more solve, as the superposition principle does not apply. That is, we can not write a solution to a nonlinear PDE as a linear combination of other solutions. This eliminates our ability to use eigenfunctions and transform methods, so we must rely on the method of characteristics to arrive at a solution. The two phenomena that this paper primarily focuses on are rarefaction waves and shock waves, which, in their simplest form, can be modeled by the first-order nonlinear equation

\[ u_t + a(u)u_x = 0. \tag{1} \]

Although both phenomena utilize this equation as their basic model, there is a key difference in the physical feature of each wave. Rarefaction waves occur when there is a decompression of the medium, whereas a shock wave occurs when there is a compression of the medium. These conditions arise from the speed of the wave fronts created initially by the disturbance. If the leading wave front is travelling faster than the trailing wave front, then an increasing gap is created between them, causing decompression. If the leading wave front is travelling slower, then the trailing wave front "overtakes" the leading wave front and causes compression. These features are determined by the initial conditions of the model given for a PDE, as well as the function \( a(u) \), which we will discuss throughout this paper. This paper will demonstrate how to use the method of characteristics to produce solutions and graphically depict rarefaction and shock waves.
Rarefaction Waves

Both rarefaction and shock waves are modeled by nonlinear hyperbolic PDEs, the simplest of which is given by (1). To demonstrate how one would solve this PDE, let us first consider \( a \) as a function of \( x \) and \( t \). By using the method of characteristics, we take characteristic curves to this PDE as solutions to the ordinary differential equation (ODE), \( \frac{dx}{dt} = a(x, t) \). Every point \((x_0, t_0)\) has a unique characteristic curve passing through it, and we can calculate that as follows by

\[
0 = u_x a(x, t) + u_t = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} = \frac{du}{dt} \tag{2}
\]

Therefore, \( u(x, t) \) is constant along each characteristic curve, or \( u(x, t) = u(x_0, t_0) \). With this property established, let us solve the PDE when, for example, \( a(x, t) = e^{x+t} \), which has the characteristic equation \( \frac{dx}{dt} = e^{x+t} \). The ODE can be separated into the form \( e^{-x} dx = e^t dt \), which has solutions given by \( e^{-x} = -e^t + C \), where \( C \) is an arbitrary constant. Since \( u(x, t) \) is constant along each curve, we can write the solution as a function of \( C \), which is given by

\[
u(x, t) = f(C) = f(e^{-x} + e^t). \]

Now let us take \( a \) as function of \( u \). In this case, \( u \) is still constant along each characteristic curve, which can be shown through the same calculations as (2). In particular, let us explore the equation

\[
u_t + uu_x = 0 \tag{3}
\]

This is known as the inviscid Burger’s equation, which is a very basic equations for fluids. Now the characteristic equation is given by \( \frac{dx}{dt} = u(x, t) \), which means each solution to (3) will produce a different family of characteristic curves. From (2), we already know that \( u \) is constant on each characteristic curve, hence \( \frac{dx}{dt} \) is a constant as well. Three principles arise from these observations:

(a) Each characteristic curve is a straight line. So each solution \( u(x, t) \) has a family of straight lines (of various slopes) as its characteristics.
(b) The solution is constant on each such line

(c) The slope of each such line is equal to the value of $u(x,t)$ on it. \(^1\)

Now let’s impose the initial condition $u(x,0) = \phi(x)$. By principal (c), the slope of the characteristic line going through some point $(x_0,0)$ will equal $\phi(x_0)$. The problem that arises, then, is when two characteristic lines intersect, as that means $u$ and $\phi(x)$ become multi-valued. Figure 1 represents this scenario, where two characteristic lines, one going through $(x_0,0)$ and the other through $(x_1,0)$, intersect at some point $(x,t)$. At this point, $u = \phi(x_0) = \phi(x_1)$, but this is impossible because the lines have different slopes.

There are three different ways of getting around this dilemma. The first is to avoid any intersection whatsoever by assuming that $\phi(x)$ never decreases for $t \geq 0$, and this leads to rarefaction waves. The second is to allow discontinuities in the solution. This leads into the theory of shock waves, which we will cover later. Lastly, we could just assume that the solution exists only near $t = 0$, and that the solution breaks down when it departs from this line. However, for the purposes of this paper, we will only focus on the first two methods.

Let’s start by using the method of assuming $\phi(x)$ never decreases to solve the general equation (1) with intial condition $u(x,0) = \phi(x)$. Since none of the characteristic lines intersect, we can write the slope of any characteristic line going through $(z,0)$ and $(x,t)$ as

$$\frac{x-z}{t-0} = \frac{dx}{dt} = a(u(x,t)) = a(u(z,0)) = a(\phi(z))$$

as shown in Figure 2.

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\(^1\)Principles (a), (b), and (c) taken directly from Strauss
This implies that $u(x, t) = \phi(z)$, where $z$ satisfies the equation $x - z = ta(\phi(z))$. Thus, $z$ is a function of $x$ and $t$, and we can write the solution to $u$, explicitly, as

$$u(x, t) = \phi(z(x, t))$$

(4)

Since we are imposing the condition that none of the characteristic lines intersect for $t \geq 0$, we must make sure the slope is increasing. That is:

$$a(\phi(z)) \leq a(\phi(w)) \quad \text{for } z \leq w.$$ 

Thus, we arrive at the solution to the rarefaction wave modeled by (1). To represent the imposed condition visually, let us take the inviscid Burger’s equation with particular initial condition

$$\phi(x) = \begin{cases} 
1 & \text{if } x < 0 \\
2 & \text{if } x > 0 
\end{cases}$$

(5)

The slopes of the characteristic lines are then given by $\frac{dx}{dt} = \phi(x(t)) = \phi(x(0))$, which we illustrate on a $t$ vs. $x$ graph in Figure 3.
Figure 3

The solid lines in this figure represent each of the characteristic curves, and we can see that the waves that originate from $x(0) < 0$ have greater speed than those originating from $x(0) > 0$. Thus, a gap is created, represented by the dotted lines, which grows larger in width as $t$ increases. Now that we have successfully modeled rarefaction waves through (1), let us turn our attention to shock waves, where we take into account the intersection of characteristic curves by allowing discontinuities.

**Discontinuity and Shock Waves**

In our discussion of rarefaction waves we defined the occurrence as a fast wave traveling from left to right preceding a slower wave, thus creating non-intersecting parallel characteristic lines. Now we have the peculiar case where the characteristic lines are not parallel and eventually cross one another. We must now refer back to our previously stated principal which states that the value of the solution at a particular point lying on a unique characteristic line is equal to the slope of that particular line, thus arriving at the situation where the solution to a shock wave system acquires multiple values corresponding to each line at the crossing point. Herein lies the dilemma of shock waves, where we understand the solution to be representative of a physical quantity such as velocity or pressure, where it must assume a unique value at each point on the characteristic line; however, the mathematical model we have developed breaks down and the solutions are no longer uniquely determined.

In order to get a sense of the visual interpretation we refer back to Figure 3 where we
observed the characteristic lines emerging from the x-axis, propelling out to the right of the plane and never crossing each other for all \( t \geq 0 \), consequently providing well defined solutions for future values of time. Figure 4 below, illustrates the case where the initial data decreases and the characteristic lines initially do not cross for lines originating from \( x(0) < 0 \) but eventually reach a critical point in time where the lines meet.

![Figure 4](image)

Expanding on the figure, let's take the inviscid Burger's equation with the following initial condition:

\[
\phi(x) = \begin{cases} 
2 & \text{if } x < 0 \\
1 & \text{if } x > 0 
\end{cases}.
\]

These initial conditions differ from the previous conditions posed for the rarefaction waves such that the initial condition is greater for \( x < 0 \) than \( x > 0 \). The jump discontinuity at \( x = 0 \) implies that the lines have a smaller slope for \( x(0) < 0 \) and greater slopes for \( x(0) > 0 \), thereby revisiting our previous claim that shock waves take the form of the leading wave front traveling slower than the trailing wave front. The decreasing initial condition then provides us with a region in the plane where the intersection of the characteristic lines with different slopes, produce multiple solutions.

The big question now is what occurs in our inviscid Burger's equation for \( x = 0 \), and the answer to that is the occurrence of a jump discontinuity where the solution abruptly
changes value. In other words we can observe our initial condition having the form of a step function at the origin, where the observed discontinuity is represented as the shock wave. To demonstrate how one would solve this PDE, let $a(u) = A'(u)$ in equation (1), thereby allowing the equation to be rewritten as:

$$u_t + A(u)_x = 0. \tag{7}$$

We now require equation (7) to hold in the sense of distributions, such that both sides are the same distribution and thereby satisfy the PDE

$$\int_0^\infty \int_{-\infty}^{\infty} [u \psi_t + A(u) \psi_x] \, dx \, dt = 0 \tag{8}$$

for all test functions $\psi(x,t)$. A test function is one such that all the derivatives of the function exist and vanish outside the defined bounds. Keep in mind that a solution of this type satisfying our PDE is referred to as a weak solution because although all of its derivatives may not exist we have defined a case for where it satisfies the equation. We now make the conjecture that the shock occurs along the curve $x = \zeta(t)$, see Figure 5 below. Because the shock occurs at the jump discontinuity of our step function (equation 6), the left and right limits are defined as $u^- (t) = u(x^-, t)$ and $u^+ (t) = u(x^+, t)$ respectively.

![Figure 5](image-url)
By taking equation (8) and splitting the inner integral into two separate definite integrals with the following limits, \(-\infty\) to \(\xi(t)\) and \(\xi(t)\) to \(+\infty\), and applying the derivative of distribution we obtain the following

\[
\int_0^\infty \int_{-\infty}^\infty [u\psi_t + A(u)\psi_x] \, dx \, dt = 0 \\
\int_0^\infty \int_{-\infty}^{\xi(t)} [-u_t\psi - A(u)\psi] \, dx \, dt + \int_0^\infty \int_{\xi(t)}^{\infty} [-u_t\psi - A(u)\psi] \, dx \, dt = 0.
\]

Now if we consider the integrand of the separated integrals as the divergence of some vector field \(\vec{F}\), we recover the vector field to be \(\vec{F} = (-u\psi, -A(u)\psi)\). By the divergence theorem, we can rewrite each double integral as a single integral over the line \(x = \xi(t)\), where the integrands are given by the dot product of the vector \(\vec{F}\) and \(\vec{n}\), \(\vec{n}\) being the outward unit normal vector of the shock curve. Taking into account the left and right limits as seen in Figure (5), we write the above equation as:

\[
\int_{x=\xi(t)} [u^+\psi_n + A(u^+)\psi_n] \, dl + - \int_{x=\xi(t)} [u^-\psi_n + A(u^-)\psi_n] \, dl = 0 \\
\int_{x=\xi(t)} [u^+\psi_n + A(u^+)\psi_n] \, dl = \int_{x=\xi(t)} [u^-\psi_n + A(u^-)\psi_n] \, dl = 0
\]

By setting the integrands equal to one another and applying some minor algebra we get the following result

\[
\frac{A(u^+) - A(u^-)}{u^+ - u^-} = -\frac{n_t}{n_x} = s(t).
\] (9)

Here, the arbitrary test function \(\psi(x, t)\) is cancelled from each side of the above equation and is set equal to \(s(t)\), the speed of the shock and also the reciprocal of the slope in figure (5) with respect to the change in corresponding axes on the graph. Equation (9) above is also known as the Rankine-Hugoniot formula for the speed of the shock wave. We have required for a shock wave to exist, the characteristic lines must intersect and the characteristic behind the shock must have a greater slope than the characteristic after the shock. Mathematically
this is represented by the following result:

\[ a(u^+) < s(t) < a(u^-) \]  \hspace{1cm} (10)

At the occurrence of a shock wave for our model in equation (1) we have a discontinuity, which we have shown must satisfy both equation (9) and equation (10).

In exploring the two phenomena that are rarefaction waves and shock waves, we arrived at the method of characteristics to find solutions for the first-order nonlinear equation \( u_t + a(u)u_x = 0 \). First we illustrated how the two differed. Rarefaction waves occur when there is a decompression of the medium and shock waves occur when the medium experiences a compression. Our graphical interpretations along with our initial conditions for the inviscid Burger equation furthered our arguments by showing how the slope/speed of the characteristic lines determined the presence of the phenomena. Using the derivative of distribution and the divergence theorem we managed to discover how the existence of a shock waves depends on the characteristic lines intersecting and the characteristic behind the shock having a greater slope than the characteristic after the shock. Shock waves are represented by a jump discontinuity that satisfies Rankine-Hugoniot’s formula and although they are a product of multiple values, they both exist in the physical world and could be solved mathematically.
References
