Waves and Huygens’s Principle

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April 30, 2014
Introduction

We studied the wave equation and Huygens’s principle in higher dimensions. It turns out that Huygens’s principle is valid for any vector space with an odd number of spatial dimensions, so that the value at a point \((x_o, t_o)\) can be evaluated by integrating over the domain of dependence. However, Huygens’s principle is not valid for even spatial dimensions. Although we can use the method of descent to reduce it to a solvable problem in odd spatial dimensions, we must integrate over the entire region to determine the value of a point \((x_o, t_o)\).
Domain of Influence and Domain of Dependence in 1D

Recall that for any particular initial value problem in 1 spatial dimension, an initial condition at the point \((x_0, t_0)\) affects the solution for \(t \geq t_0\) over the set of points enclosed by the lines \(x + ct\) and \(x - ct\). This is because a disturbance travels in both directions at the wave speed \(c\). The set of points influenced by the initial condition is called \textit{the domain of influence} of point \((x_0, t_0)\). The ‘inverse’ is also true. If we choose a point \((x, t)\) for \(t > 0\), then by tracing backwards in time in both directions along lines of slope \(c\), we find that \(u(x,t)\) depends only on the values of \(\phi\) at the two points \(x + ct\) and \(x - ct\), and on the values of \(\psi\) within the interval \([x - ct, x + ct]\). We therefore say that the interval \((x + ct, x - ct)\) is the \textit{interval of dependence} of the point \((x, t)\) on \(t = 0\). The entire triangle is often called the \textit{domain of dependence} or the \textit{past history} of the point \((x, t)\).

The solution to the wave equation in 1D is given by d’Alembert’s:

\[
   u(x, t) = \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x - ct}^{x + ct} \psi(s) ds
\]
In 2 Spatial Dimensions

If we are to imagine a wave and its domains of influence and dependence over 2 spatial dimensions, it is helpful to imagine a disturbance like a drop falling onto the surface of a body of water. The waves resulting from the disturbance travel in all directions away from the drop, which we can think of as the initial condition \((x_0, t_0)\). Think of concentric ripples traveling away from the initial disturbance at wave speed \(\leq c\). Then the domain of influence is a cone bounded by lines of slope \(c\). Similarly, if we fix any point \((x, t)\), then its past history is a cone backward in time. The solution to the 2 dimensional wave equation is:

\[
\begin{align*}
  u(x_0, y_0, t_0) &= \frac{1}{2\pi c} \int \int_D \frac{\psi(x, y)}{\sqrt{c^2 t_0^2 - (x - x_0)^2 - (y - y_0)^2}} \, dx \, dy \\
  &\quad + \frac{\partial}{\partial t_0} \frac{1}{2\pi c} \int \int_D \frac{\phi(x, y)}{\sqrt{c^2 t_0^2 - (x - x_0)^2 - (y - y_0)^2}} \, dx \, dy \\
  \text{where } D \text{ is the disc } (x - x_0)^2 + (y - y_0)^2 \leq c^2 t_0^2
\end{align*}
\]
In 3 Spatial Dimensions, and Higher Spatial Dimensions

The domain of influence and dependence is a sphere for an initial value problem with 3 spatial dimensions. In higher dimensions, we can think of the domain of influence and dependence as a hyper-sphere.

Kirchhoff’s formula gives the solution to the wave equation in 3 spatial dimensions:

$$u(x, t_0) = \frac{1}{4\pi c^2 t_0^2} \int \int_S \psi(x) dS + \frac{\partial}{\partial t_0} \left[ \frac{1}{4\pi c^2 t_0^2} \int \int_S \phi(x) dS \right]$$

where $S$ is the sphere of center $x_0$ and radius $ct_0$. 
Domain of Dependence and Huygens’s Principle

“The domain of dependence of the point \((x_o, t_o)\) on the solution \(U(x_o, t_o)\) of a hyperbolic PDE is the subset of the initial conditions which uniquely determine the value \(U(x_o, t_o)\).”

Huygens’s principle tells us that for n-odd, we can evaluate the function at a point \((x_o, t_o)\) by integrating over the boundary of the domain of dependence.
Huygens’s Principle describes the domain of influence for PDE’s in odd dimensions to be a hyper-sphere. We use the method of spherical means to obtain the solution to the wave equation in odd dimensions:

\[
\begin{align*}
u(x, t) &= \frac{1}{\gamma_n} \left( \frac{\partial}{\partial t} \right) \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left( t^{n-2} \int_{\partial B(x,t)} \phi(y) dS(y) \right) \\
&\quad + \frac{1}{\gamma_n} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} \left( t^{n-2} \int_{\partial B(x,t)} \psi(y) dS(y) \right)
\end{align*}
\]

where \( \gamma_n = 1 \cdot 3 \cdot 5 \cdots (n - 2) \)
The Solution to the Wave Equation in Even Dimensions

Use the method of descent to effectively reduce the problem in even dimensions to a problem in odd dimensions by setting one of the coordinates $\equiv 0$. Using the method of descent, we find that the solution to the wave equation in even dimensions is:

$$u(x, t) = \frac{1}{\gamma_n} \left( \frac{\partial}{\partial t} \right) \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \oint_{B(x,t)} \frac{\phi(y)}{\left( t^2 - |y - x|^2 \right)^{1/2}} dy \right)$$

$$+ \frac{1}{\gamma_n} \left( \frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} \left( t^n \oint_{B(x,t)} \frac{\psi(y)}{\left( t^2 - |y - x|^2 \right)^{1/2}} dy \right)$$
Thus, Huygens’s principle is only valid when you have an odd number of spatial dimensions. By applying these two formulas we can solve the wave equation in any n-dimensional space, but it should be obvious by inspection that this will be faster for odd spatial dimensions (since the integral over the boundary uses less points).
