Earlier in the course we solved the Dirichlet problem for the wave equation on the finite interval $0 < x < l$ using the reflection method. This required separating the domain $(x, t) \in (0, l) \times (0, \infty)$ into different regions according to the number of reflections that the backward characteristic originating in the regions undergo before reaching the $x$ axis. In each of these regions the solution was given by a different expression, which is impractical in applications, and the method does not generalize to higher dimensions or other equations. We now study a different method of solving the boundary value problems on the finite interval, called separation of variables.

Let us start by considering the wave equation on the finite interval with homogeneous Dirichlet conditions.

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < l, \\
&u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \\
&u(0, t) = u(l, t) = 0. \end{aligned} \quad (1)$$

The idea of the separation of variables method is to find the solution of the boundary value problem as a linear combination of simpler solutions (compare this to finding the simpler solution $S(x, t)$ of the heat equation, and then expressing any other solution in terms of the heat kernel). The building blocks in this case will be the separated solutions, which are the solutions that can be written as a product of two functions, one of which depends only on $x$, and the other only on $t$, i.e.

$$u(x, t) = X(x)T(t). \quad (2)$$

Let us try to find all the separated solutions of the wave equation. Substituting (2) into the equation gives

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Dividing both sides of these identity by $-c^2 X(x)T(t)$, we get

$$-\frac{X''(x)}{X(x)} = -\frac{T''(t)}{c^2 T(t)} = \lambda. \quad (3)$$

Clearly $\lambda$ is a constant, since it is independent of $x$ from $\lambda = -T''/(c^2 T)$, and is independent of $t$ from $\lambda = -X''/X$. We will shortly see that the boundary conditions force $\lambda$ to be positive, so let $\lambda = \beta^2$, for some $\beta > 0$. One can then rewrite (3) as a pair of separate ODEs for $X(x)$ and $T(t)$

$$T'' + c^2 \beta^2 T = 0, \quad \text{and} \quad X'' + \beta^2 X = 0.$$

The solutions of these ODEs are

$$T(t) = A \cos \beta ct + B \sin \beta ct, \quad \text{and} \quad X(x) = C \cos \beta x + D \sin \beta x, \quad (4)$$

where $A$, $B$, $C$ and $D$ are arbitrary constants. From the boundary conditions in (1), we have

$$X(0)T(t) = X(l)T(t) = 0, \quad \forall t \quad \Rightarrow \quad X(0) = X(l) = 0,$$

since $T(t) \equiv 0$ would result in the trivial solution $u(x, t) \equiv 0$ (our goal is to find all separated solutions). With this boundary condition for $X(x)$, we have from (4)

$$X(0) = C = 0, \quad \text{and} \quad X(l) = D \sin \beta l = 0.$$

The solution with $D = 0$ will again lead to the trivial zero solution, so we consider the case when $\sin \beta l = 0$. But this implies that $\beta l = n\pi$ for $n = 1, 2, \ldots$, and

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l} \quad \text{for} \quad n = 1, 2, \ldots.$$
These formulas give distinct solutions for \( X(x) \), and multiplying these by the \( T(t) \) corresponding to \( \lambda_n \), we find infinitely many separated solutions

\[
 u_n(x, t) = \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \text{for } n = 1, 2, \ldots ,
\]

where \( A_n, B_n \) are arbitrary constants as before. Since a linear combination of solutions of the wave equation is also a solution, any finite sum

\[
 u(x, t) = \sum_n \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}, \quad (5)
\]

will also solve the wave equation.

Returning to our boundary value problem (1), we would like to find the solution as a linear combination of separated solutions. However, finite sums in the form (5) are very special, since not every function is a finite sum of sines and cosines. Checking the initial conditions, we have

\[
 \phi(x) = \sum_n A_n \sin \frac{n\pi x}{l},
\]

\[
 \psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}. \quad (6)
\]

Obviously, not all initial data \( \phi, \psi \) can be written as finite sums of sine functions. So instead of restricting ourselves to finite sums, we allow infinite sums, and ask the question whether any functions \( \phi, \psi \) can be written as infinite sums of sine functions. This question was first studied by Fourier, and these infinite sums have the name of Fourier series (Fourier sine series in this case). It turns out that practically any function defined on \( 0 < x < l \) can be expressed in the form (6). Leaving the question of convergence of such sums, we see that if the initial data can be expressed in the form (6), then the solution is given by (5).

The coefficients of \( t \) inside the series (5), \( \frac{n\pi c}{l} \), are called the frequencies. For a violin string of length \( l \), we had \( c^2 = \frac{T}{\rho} \), so the frequencies are

\[
 \frac{n\pi \sqrt{T}}{l\sqrt{\rho}} \quad n = 1, 2, \ldots
\]

The smallest frequency, \( \frac{\pi \sqrt{T}}{l\sqrt{\rho}} \), is the fundamental note, while the double, triple, and so on of the fundamental note are the overtones. Notice that by shortening the length \( l \) of the vibrating portion of the string with a finger, a violinist produces notes of higher frequency.

### 18.1 Heat equation

For the Dirichlet heat problem on the finite interval,

\[
 \left\{ \begin{array}{l}
 u_t - ku_{xx} = 0, \quad \text{for } 0 < x < l, \\
 u(x, 0) = \phi(x), \\
 u(0, t) = u(l, t) = 0,
 \end{array} \right. \quad (7)
\]

we similarly search for all the separated solutions in the form \( u(x, t) = X(x)T(t) \). In this case the equation gives

\[
 \frac{X''}{X} = \frac{T'}{kt} = \beta^2,
\]

and the resulting ODEs are

\[
 T' = -\beta^2 kT, \quad \text{and} \quad X'' + \beta^2 X = 0.
\]
The solution for the $T$ equation is then $T(t) = Ae^{-\beta^2 kt}$, while the function $X(x)$ satisfies the same equation and boundary conditions as before. This yields the same values $\beta_n = n\pi/l$. We thus have that

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(n\pi/l\right)^2 kt} \sin \frac{n\pi x}{l}$$

is the solution to problem (7), provided that the initial data is given as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$  

Notice that as $t$ grows, all the terms in the series (8) decay exponentially, so the solution itself will decay, which makes sense in terms of heat conduction, since in the absence of a heat source, the temperatures in the rod will equalize with the zero temperature of the environment.

**Example 18.1.** Solve the following Dirichlet problem for the heat equation by separation of variables.

$$\begin{cases}
& u_t - ku_{xx} = 0, \quad 0 < x < \pi/2, \\
& u(x, 0) = 3 \sin 4x, \\
& u(0, t) = u(\pi/2, t) = 0,
\end{cases}$$

In this problem $l = \pi/2$, so $\beta_n = 2n$. We can write the initial data in the form (9),

$$3 \sin 4x = \sum_{n=1}^{\infty} A_n \sin 2nx,$$

which implies that $A_2 = 3$, and $A_n = 0$ for $n \neq 2$. But then from (8) the solution will be

$$u(x, t) = 3e^{-16kt} \sin 4x.$$  

**18.2 Eigenvalues**

The numbers $\lambda = \left(\frac{n\pi}{2}\right)^2$ are called eigenvalues, and the functions $X_n(x) = \sin \frac{n\pi x}{l}$ are called eigenfunctions. Notice that we can think of the equation $-X'' = \lambda X$ as an eigenvalue problem for the operator $-\frac{d^2}{dx^2}$ in the space of functions that satisfy the Dirichlet conditions $X(0) = X(l) = 0$. An eigenfunction is then a solution of the equation which is not identically zero, i.e. $X(x) \neq 0$.

However, unlike the operators in linear algebra, which have finitely many eigenvalues, in our case we have an infinite number of eigenvalues. This is due to the fact that the space of functions is infinite dimensional.

We return to the question of the sign of the eigenvalues. Suppose $\lambda = 0$, then we would have $X'' = 0$, which leads to $X(x) = C + Dx$. The boundary conditions then imply that $C = 0$, and $Dl = 0$, giving $X(x) \equiv 0$.

If, on the other hand, we assume that $\lambda < 0$, and write $\lambda = -\gamma^2$ for some $\gamma > 0$, then the equation for $X$ becomes $X'' = \gamma^2 X$, which has the solution

$$X(x) = Ce^{\gamma x} + De^{-\gamma x}.$$  

The boundary conditions then give

$$\begin{cases}
& C + D = 0 \\
& Ce^{\gamma l} + De^{-\gamma l} = 0
\end{cases} \Rightarrow \begin{cases}
& C = -D \\
& Ce^{2\gamma l} = C
\end{cases} \Rightarrow C = D = 0,$$

which again results in the identically zero solution $X(x) \equiv 0$. So there are no nonpositive eigenvalues.

**18.3 Conclusion**

Returning to the Dirichlet problems for the wave and heat equations on a finite interval, we solved them with the method of separation of variables. That is, we looked for the solution in the form of an infinite linear combination of separated solutions. This lead to infinite series, which solve the appropriate initial-boundary value problems, as long as the initial data can be expanded in corresponding series, which in the case of Dirichlet conditions are the Fourier sine series. The question of convergence of such series will be discussed later on, while the case of Neumann conditions will be considered in the next lecture.
The same method of separation of variables that we discussed last time for boundary problems with Dirichlet conditions can be applied to problems with Neumann, and more generally, Robin boundary conditions. We illustrate this in the case of Neumann conditions for the wave and heat equations on the finite interval.

Substituting the separated solution $u(x,t) = X(x)T(t)$ into the wave Neumann problem
\[
\begin{cases}
u_{tt} - c^2 u_{xx} = 0, & 0 < x < l, \\
u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \\
x_x(0,t) = u_x(l,t) = 0,
\end{cases}
\]
gives the same equations for $X$ and $T$ as in the Dirichlet case,
\[-X'' = \lambda X, \quad \text{and} \quad -T'' = c\lambda^2 T.
\]
However, the boundary conditions now imply
\[X'(0)T(t) = X'(l)T(t) = 0, \quad \forall t \implies X'(0) = X'(l) = 0.
\]
To find all the separated solutions, we need to find all the eigenvalues and eigenfunctions satisfying these boundary conditions. To do this, we need to consider the cases $\lambda = 0, \lambda < 0$ and $\lambda > 0$ separately.

Assume $\lambda = 0$, then the equation for $X$ is $X'' = 0$, which has the solution $X(x) = C + Dx$. The derivative is then $X'(x) = D$, and the boundary conditions imply that $D = 0$. So every constant function, $X(x) = C$, is an eigenfunction for the eigenvalue $\lambda_0 = 0$.

Next, we assume that $\lambda = -\gamma^2 < 0$, in which case the equation for $X$ takes the form
\[X'' = -\gamma^2 X.
\]
The solution to this equation is $X(x) = Ce^{\gamma x} + De^{-\gamma x}$, so $X'(x) = C\gamma e^{\gamma x} - D\gamma e^{-\gamma x}$. Checking the boundary conditions gives
\[
\begin{cases}
C\gamma - D\gamma = 0 \\
C\gamma e^{\gamma l} - D\gamma e^{-\gamma l} = 0
\end{cases} \implies
\begin{cases}
C = D \\
C\gamma(e^{2\gamma l} - 1) = 0
\end{cases} \implies C = D = 0,
\]
since $\gamma \neq 0$, and hence, also $e^{2\gamma l} - 1 \neq 0$. This leads to the identically zero solution $X(x) \equiv 0$, which means that there are no negative eigenvalues.

For the remaining case, $\lambda = \beta^2 > 0$, the equation is $X'' = -\beta^2 X$, which as we saw last time when discussing Dirichlet boundary conditions, has the solution
\[X(x) = C \cos \beta x + D \sin \beta x.
\]
The derivative of this function is
\[X'(x) = -C\beta \sin \beta x + D\beta \cos \beta x,
\]
so the boundary conditions give
\[X'(0) = D\beta = 0, \quad \text{and} \quad X'(l) = -C\beta \sin \beta l = 0.
\]
Since $\beta \neq 0$, and $C$ and $D$ cannot be both zero, we have $\sin \beta l = 0$, which implies that $\beta l = n\pi$ for $n = 1, 2, \ldots$. Then the eigenvalues and the corresponding eigenfunctions are
\[
\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos \frac{n\pi x}{l}, \quad \text{for } n = 0, 1, 2, \ldots
\]
Notice that we have also included \( n = 0 \), which gives the zero eigenvalue \( \lambda_0 = 0 \) with the eigenfunction \( X_0 = 1 \). The set of all eigenvalues, called the spectrum, for the Neumann conditions differs from that for the Dirichlet conditions by this additional eigenvalue.

In the case of \( \lambda_0 = 0 \) the \( T \) equation becomes \( T'' = 0 \), which has the solution \( T(t) = \frac{1}{2} A_0 + \frac{1}{2} B_0 t \). The factors of \( \frac{1}{2} \) are included for future convenience (to have a single formula for the Fourier coefficients).

The solutions \( T_n \) corresponding to \( \lambda_n = \left( \frac{n\pi}{l} \right)^2 \) for \( n = 1, 2, \ldots \), were found in the last lecture to be

\[
T_n(t) = A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}.
\]

Putting everything together gives the following series expansion for the solution of problem (10),

\[
u(x, t) = \frac{1}{2} A_0 + \frac{1}{2} B_0 t + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \cos \frac{n\pi x}{l},
\]

as long as the initial data can be expanded into cosine Fourier series

\[
\phi(x) = \frac{1}{2} A_0 + \sum_{n} A_n \cos \frac{n\pi x}{l},
\]

\[
\psi(x) = \frac{1}{2} B_0 + \sum_{n} \frac{n\pi c}{l} B_n \cos \frac{n\pi x}{l}.
\]

These series for the data come from plugging in \( t = 0 \) into the solution formula (12), and its derivative with respect to \( t \). We notice that in the case of the Neumann conditions we end up with cosine Fourier series for the data, while in the Dirichlet case we had sine Fourier series. This is in agreement with the reflection method, since one needs to take the odd extensions of the data in the case of Dirichlet conditions, and even extensions in the case of Neumann conditions. But odd functions (extended data) have only sines in their Fourier expansions, while even functions have only cosines.

**Example 19.1.** Solve the following Neumann problem for the wave equation by separation of variables.

\[
\begin{aligned}
&u_{tt} - 4u_{xx} = 0, \quad \text{for } 0 < x < \pi, \\
u(x, 0) = 3 \cos x, \quad &u_t(x, 0) = 1 - \cos 4x \\
u_x(0, t) = u_x(\pi, t) = 0,
\end{aligned}
\]

In this problem \( l = \pi \), so \( \beta_n = n \). Notice also that \( c = 2 \), and we can write the initial data in the form (13) as follows,

\[
3 \cos x = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos nx,
\]

\[
1 - \cos 4x = \frac{1}{2} B_0 + \sum_{n=1}^{\infty} 2n B_n \cos nx.
\]

These identities imply that \( A_1 = 3 \), and \( A_n = 0 \) for \( n \neq 1 \), and \( B_0 = 2, B_4 = -\frac{1}{8} \), and \( B_n = 0 \) for \( n \neq 0, 4 \). Then solution (12) will take the form

\[
u(x, t) = t + 3 \cos 2t \cos x - \frac{1}{8} \sin 8t \cos 4x.
\]
19.1 Heat equation

For the Neumann heat problem on the finite interval,

\[
\begin{align*}
\frac{u_t - ku_{xx}}{} &= 0, \quad \text{for } 0 < x < l, \\
u(x,0) &= \phi(x), \\
u_x(0,t) &= \nu_x(l,t) = 0,
\end{align*}
\]

the equations for \(X\) and \(T\) factors of the separated solution \(u(x,t) = X(x)T(t)\) are

\[
X'' = -\lambda X, \quad \text{and} \quad T' = -\lambda kT.
\]

The boundary conditions are the same as in the wave problem (10), so one gets the same eigenvalues and eigenfunctions (11). For the eigenvalue \(\lambda_0 = 0\), the \(T\) equation is \(T' = 0\), so \(T_0(t) = \frac{1}{2}A_0\). For the positive eigenvalues we found the solutions for \(T\) in the last lecture to be

\[
T_n(t) = A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}.
\]

Thus, the solution to the heat Neumann problem is given by the series

\[
u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos \frac{n\pi x}{l},
\]

as long as the initial data can be expanded into the cosine Fourier series

\[
\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.
\]

19.2 Mixed boundary conditions

Sometimes one needs to consider problems with mixed Dirichlet-Neumann boundary conditions, i.e. Dirichlet conditions at one end of the finite interval, and Neumann conditions at the other. Examples of such problems are vibrations of a finite string with one free and one fixed end, and the heat conduction in a finite rod with one insulated end, and the other end kept at a constant zero temperature.

In such cases the method of separation of variables leads to the eigenvalue problem

\[
\begin{align*}
X'' &= -\lambda X, \\
X(0) &= X'(l) = 0,
\end{align*} \quad \text{or} \quad \begin{align*}
X'' &= -\lambda X, \\
X'(0) &= X(l) = 0.
\end{align*}
\]

One can then show that the eigenvalues are \(\lambda_n = \left(\frac{n + \frac{1}{2}}{l}\right)^2\), and the corresponding eigenfunctions for the respective problems are

\[
X_n(x) = \sin \frac{(n + \frac{1}{2}) \pi x}{l}, \quad \text{and} \quad X_n(x) = \cos \frac{(n + \frac{1}{2}) \pi x}{l}, \quad \text{for } n = 0, 1, 2, \ldots,
\]

which is left as a homework exercise.

19.3 Conclusion

Similar to the case of the Dirichlet problems for heat and wave equations, the method of separation of variables applied to the Neumann problems on a finite interval leads to an eigenvalue problem for the \(X(x)\) factor of the separated solution. In this case, however, we discovered a new eigenvalue \(\lambda = 0\) in addition to the eigenvalues found for the Dirichlet problems. Then the general solutions of the Neumann problems for wave and heat equations can be written in series forms, as (infinite) linear combinations of all separated solutions, as long as the initial data can be expanded in cosine Fourier series. We will discuss in detail the questions on whether and how a given function can be expanded into Fourier series later on.
To solve the boundary value problems for the heat and wave equations on the finite interval \((0, l)\), we used the method of separation of variables. This method relies on the idea of representing the solution to the general boundary value problem as a linear combination of separated solutions \(u(x, t) = X(x)T(t)\). For these separated solutions the PDEs reduce to pairs of ODEs. Indeed, plugging the separated solution into the heat and wave equations, and separating the variables gives the following.

**Wave equation**, \(u_{tt} - c^2u_{xx} = 0\)

\[
T''(t)X(x) - c^2T(t)X''(x) = 0
\]

\[
\frac{X''}{X} = -\frac{T''}{c^2T} = \lambda
\]

\[X'' = -\lambda X \quad \text{and} \quad T'' = -\lambda c^2 T.\]

**Heat equation**, \(u_t - ku_{xx} = 0\)

\[
T'(t)X(x) - kT(t)X''(x) = 0
\]

\[
\frac{X''}{X} = -\frac{T'}{kT} = \lambda
\]

\[X'' = -\lambda X \quad \text{and} \quad T' = -\lambda kT.\]

The different boundary conditions applied to the separated solution imply the following.

**Dirichlet**: \(u(0, t) = u(l, t) = 0 \Rightarrow X(0)T(t) = X(l)T(t), \forall t \Rightarrow X(0) = X(l) = 0.\)

**Neumann**: \(u_x(0, t) = u_x(l, t) = 0 \Rightarrow X'(0)T(t) = X'(l)T(t), \forall t \Rightarrow X'(0) = X'(l) = 0.\)

Using these conditions for the \(X(x)\) component of the separated solution, we arrive at the following eigenvalue problems

**Dirichlet**: \[
\begin{cases}
X'' = -\lambda X, \\
X(0) = X(l) = 0.
\end{cases}
\]

**Neumann**: \[
\begin{cases}
X'' = -\lambda X, \\
X'(0) = X'(l) = 0.
\end{cases}
\]

The values of \(\lambda\) for which these problems have a nontrivial solution \((X \neq 0)\) are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions. Notice that for each of the cases \(\lambda < 0, \lambda = 0, \lambda > 0\) the equation \(X'' = -\lambda X\) has qualitatively different solutions due to the sign of the discriminant of the characteristic quadratic equation. After considering the three cases separately, and eliminating those which lead to only trivial solutions, we arrive at the following eigenvalues and eigenfunctions for the respective boundary conditions.

**Dirichlet**: \[
\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin \frac{n\pi x}{l}
\]

for \(n = 1, 2, \ldots\)

**Neumann**: \[
\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \cos \frac{n\pi x}{l}
\]

for \(n = 0, 1, 2, \ldots\)

Notice that the eigenvalues for the Dirichlet and Neumann boundary conditions are exactly the same, with the only extra eigenvalue \(\lambda_0 = 0\) in the Neumann case. The eigenfunctions corresponding to these eigenvalues are sine in the Dirichlet case, and cosine in the Neumann case. This corresponds to the necessity of taking odd extensions of the Dirichlet data, and even extension of the Neumann data, which we saw in the reflection method.

Using the above eigenvalues \(\lambda_n\), we can solve the \(T\) equations corresponding to the wave and heat equations, arriving at the following solutions.
Wave:
\[ T_n(t) = A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l}, \]
for \( n = 1, 2, \ldots \)
\[ T_0(t) = \frac{A_0}{2} + \frac{B_0}{2} t. \]

Heat:
\[ T_n(t) = A_n e^{-(n\pi/l)^2 kt}, \]
for \( n = 1, 2, \ldots \)
\[ T_0(t) = \frac{A_0}{2}. \]

Then the solutions to the boundary value problems can be written as linear combinations of the separated solutions \( u_n(x,t) = X_n(x)T_n(t) \), leading to the series solutions for the wave and heat boundary value problems. These solutions will solve the corresponding problems, if the initial conditions are also satisfied, which will be the case provided the initial data can be expanded into corresponding Fourier sine and cosine series. We next list each of these solutions for the wave and heat boundary value problems.

**Wave Dirichlet:**
\[ \left\{ \begin{array}{l}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < l, \\
u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \\
u(0,t) = u(l,t) = 0.
\end{array} \right. \]

The series solution has the form
\[ u(x,t) = \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}, \]
provided the initial data can be expanded into the Fourier sine series
\[ \phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}, \]
\[ \psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}. \]

**Wave Neumann:**
\[ \left\{ \begin{array}{l}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for } 0 < x < l, \\
u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \\
\frac{\partial u}{\partial x}(0,t) = u_x(l,t) = 0.
\end{array} \right. \]

The series solution has the form
\[ u(x,t) = \frac{A_0}{2} + \frac{B_0}{2} t + \sum_{n=1}^{\infty} \left( A_n \cos \frac{n\pi ct}{l} + B_n \sin \frac{n\pi ct}{l} \right) \cos \frac{n\pi x}{l}, \]
provided the initial data can be expanded into the Fourier cosine series
\[ \phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}, \]
\[ \psi(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos \frac{n\pi x}{l}. \]
Heat Dirichlet:

\[
\begin{cases}
  u_t - ku_{xx} = 0 & \text{for } 0 < x < l, \\
  u(x, 0) = \phi(x), \\
  u(0, t) = u(l, t) = 0.
\end{cases}
\]

The series solution has the form

\[
u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin \left(\frac{n\pi x}{l}\right),
\]

provided the initial data can be expanded into the Fourier sine series

\[
\phi(x) = \sum_{n=1}^{\infty} A_n \sin \left(\frac{n\pi x}{l}\right).
\]

Heat Neumann:

\[
\begin{cases}
  u_t - ku_{xx} = 0 & \text{for } 0 < x < l, \\
  u(x, 0) = \phi(x), \\
  u_x(0, t) = u_x(l, t) = 0.
\end{cases}
\]

The series solution has the form

\[
u(x, t) = A_0 \frac{1}{2} + \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \cos \left(\frac{n\pi x}{l}\right),
\]

provided the initial data can be expanded into the Fourier cosine series

\[
\phi(x) = A_0 \frac{1}{2} + \sum_{n=1}^{\infty} A_n \cos \left(\frac{n\pi x}{l}\right).
\]

Having these series solutions, all there remains to do to find the complete solutions to the boundary value problems is to compute the coefficients in the Fourier expansions of the initial data. This will be the subject of our subsequent lectures, in which we will also address questions of convergence of such series.