In previous lectures we completely solved the initial value problem for the heat equation on the whole line, i.e. in the absence of boundaries. Next, we turn to problems with physically relevant boundary conditions. Let us first add a boundary consisting of a single endpoint, and consider the heat equation on the half-line $D = (0, \infty)$. The following initial/boundary value problem, or IBVP, contains a Dirichlet boundary condition at the endpoint $x = 0$.

\[
\begin{aligned}
&v_t - kv_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\
&v(x, 0) = \phi(x), & x > 0, \\
&v(0, t) = 0, & t > 0.
\end{aligned}
\]

(1)

If the solution to the above mixed initial/boundary value problem exists, then we know that it must be unique from an application of the maximum principle. In terms of the heat conduction, one can think of $v$ in (1) as the temperature in an infinite rod, one end of which is kept at a constant zero temperature. The initial temperature of the rod is then given by $\phi(x)$.

Our goal is to solve the IBVP (1), and derive a solution formula, much like what we did for the heat IVP on the whole line. But instead of constructing the solution from scratch, it makes sense to try to reduce this problem to the IVP on the whole line, for which we already have a solution formula. This is achieved by extending the initial data $\phi(x)$ to the whole line. We have a choice of how exactly to extend the data to the negative half-line, and one should try to do this in such a fashion that the boundary condition of (1) is automatically satisfied by the solution to the IVP on the whole line that arises from the extended data. This is the case, if one chooses the odd extension of $\phi(x)$, which we describe next.

By definition, a function $\psi(x)$ is odd, if $\psi(-x) = -\psi(x)$. But then plugging in $x = 0$ into this definition, one gets $\psi(0) = 0$ for any odd function. Recall also that the solution $u(x, t)$ to the heat IVP with odd initial data is itself odd in the $x$ variable. This follows from the fact that the sum $[u(x, t) + u(-x, t)]$ solves the heat equation and has zero initial data, hence, it is the identically zero function by the uniqueness of solutions. Then, by our above observation for odd functions, we would have that $u(0, t) = 0$ for any $t > 0$, which is exactly the boundary condition of (1).

This shows that if one extends $\phi(x)$ to an odd function on the whole line, then the solution with the extended initial data automatically satisfies the boundary condition of (1). Let us then define

\[
\phi_{\text{odd}}(x) = \begin{cases} 
\phi(x) & \text{for } x > 0, \\
-\phi(-x) & \text{for } x < 0, \\
0 & \text{for } x = 0.
\end{cases}
\]

(2)

It is clear that $\phi_{\text{odd}}$ is an odd function, since we defined it for negative $x$ by reflecting the $\phi(x)$ with respect to the vertical axis, and then with respect to the horizontal axis. This procedure produces a function whose graph is symmetric with respect to the origin, and thus it is odd. One can also verify this directly from the definition of odd functions. Now, let $u(x, t)$ be the solution of the following IVP on the whole line

\[
\begin{aligned}
&u_t - ku_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\
&u(x, 0) = \phi_{\text{odd}}(x).
\end{aligned}
\]

(3)

From previous lectures we know that the solution to (3) is given by the formula

\[
u(x, t) = \int_{-\infty}^{\infty} S(x-y, t) \phi_{\text{odd}}(y) \, dy, \quad t > 0.
\]

(4)

Restricting the $x$ variable to only the positive half-line produces the function

\[
v(x, t) = u(x, t)|_{x \geq 0}.
\]

(5)

We claim that this $v(x, t)$ is the unique solution of IBVP (1). Indeed, $v(x, t)$ solves the heat equation on the positive half-line, since so does $u(x, t)$. Furthermore,

\[
v(x, 0) = u(x, 0)|_{x > 0} = \phi_{\text{odd}}(x)|_{x > 0} = \phi(x),
\]
and \( v(0, t) = u(0, t) = 0 \), since \( u(x, t) \) is an odd function of \( x \). So \( v(x, t) \) satisfies the initial and boundary conditions of (1).

Returning to formula (4), we substitute the expressions for \( \phi_{\text{odd}} \) from (2) and write
\[
 u(x, t) = \int_{-\infty}^{\infty} S(x - y, t) \phi_{\text{odd}}(y) \, dy + \int_{-\infty}^{0} S(x - y, t) \phi_{\text{odd}}(y) \, dy
\]
\[
= \int_{0}^{\infty} S(x - y, t) \phi(y) \, dy - \int_{-\infty}^{0} S(x - y, t) \phi(-y) \, dy.
\]
Making the change of variables \( y \mapsto -y \) in the second integral on the right, and flipping the integration limits gives
\[
u(x, t) = \int_{0}^{\infty} S(x - y, t) \phi(y) \, dy - \int_{0}^{\infty} S(x + y, t) \phi(y) \, dy.
\]

Using (5) and the above expression for \( u(x, t) \), as well as the expression of the heat kernel \( S(x, t) \), we can write the solution formula for the IBVP (1) as follows
\[
v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[ e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \right] \phi(y) \, dy.
\] (6)

The method used to arrive at this solution formula is called the method of odd extensions or the reflection method.
We can make a physical sense of formula (6) by interpreting the integrand as the contribution from the point \( y \) minus the heat loss from this point due to the constant zero temperature at the endpoint.

**Example 12.1.** Solve the IBVP (1) with the initial data \( \phi(x) = e^x \).

Using the solution formula (6), we have
\[
v(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[ e^{-(x-y)^2/4kt} e^y - e^{-(x+y)^2/4kt} e^y \right] \, dy.
\] (7)

Combining the exponential factors of the first product under the integral, we will get an exponential with the following exponent
\[
\frac{-y^2 - 2(x + 2kt)y + x^2}{4kt} = -\left( y - \frac{x + 2kt}{\sqrt{4kt}} \right)^2 + kt + x = -p^2 + kt + x,
\]
where we made the obvious notation
\[
p = \frac{y - x - 2kt}{\sqrt{4kt}}.
\]

Similarly, the exponent of the combined exponential from the second product under integral (7) is
\[
\frac{-y^2 + 2(x - 2kt)y + x^2}{4kt} = -\left( y + x - 2kt \right)^2 + kt - x = -q^2 + kt - x,
\]
with
\[
q = \frac{y + x - 2kt}{\sqrt{4kt}}.
\]

Braking integral (7) into a difference of two integrals, and making the changes of variables \( y \mapsto p \), and \( y \mapsto q \) in the respective integrals, we will get
\[
v(x, t) = e^{kt+x} \frac{1}{\sqrt{\pi}} \int_{-\frac{2kt}{\sqrt{4kt}}}^{\infty} e^{-p^2} \, dp - e^{kt-x} \frac{1}{\sqrt{\pi}} \int_{\frac{2kt}{\sqrt{4kt}}}^{\infty} e^{-q^2} \, dq.
\] (8)
Notice that
\[
\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} dy = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y^2} dy + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-y^2} dy = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{x + 2kt}{\sqrt{4kt}} \right),
\]
and similarly for the second integral. Putting these back into (8), we will arrive at the solution
\[
v(x, t) = e^{kt+x} \left[ \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{x + 2kt}{\sqrt{4kt}} \right) \right] - e^{kt-x} \left[ \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{x - 2kt}{\sqrt{4kt}} \right) \right].
\]

\[\square\]

### 12.1 Neumann boundary conditions

Let us now turn to the Neumann problem on the half-line,
\[
\begin{align*}
  w_t - kw_{xx} &= 0, & 0 < x < \infty, 0 < t < \infty, \\
  w(x, 0) &= \phi(x), & x > 0 \\
  w_x(0, t) &= 0, & t > 0.
\end{align*}
\]

To find the solution of (9), we employ a similar idea used in the case of the Dirichlet problem. That is, we seek to reduce the IBVP to an IVP on the whole line by extending the initial data \( \phi(x) \) to the negative half-axis in such a fashion that the boundary condition is automatically satisfied.

Notice that if \( \psi(s) \) is an even function, i.e. \( \psi(-x) = \psi(x) \), then its derivative function will be odd. Indeed, differentiating in the definition of the even function, we get \( -\psi'(-x) = \psi'(x) \), which is the same as \( \psi'(-x) = -\psi'(x) \). Hence, for an arbitrary even function \( \psi(x) \), \( \psi'(0) = 0 \). It is now clear that extending the initial data so that the resulting function is even will produce solutions to the IVP on the whole line that automatically satisfy the Neumann condition of (9).

We define the even extension of \( \phi(x) \),
\[
\phi_{\text{even}} = \begin{cases} 
  \phi(x) & \text{for } x \geq 0, \\
  \phi(-x) & \text{for } x \leq 0,
\end{cases}
\]
and consider the following IVP on the whole line
\[
\begin{align*}
  u_t - ku_{xx} &= 0, & -\infty < x < \infty, 0 < t < \infty, \\
  u(x, 0) &= \phi_{\text{even}}(x).
\end{align*}
\]

It is clear that the solution \( u(x, t) \) of the IVP (11) will be even in \( x \), since the difference \( [u(-x, t) - u(x, t)] \) solves the heat equation and has zero initial data. We then use the solution formula for the IVP on the whole line to write
\[
u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi_{\text{even}}(y) dy, \quad t > 0,
\]
and take
\[
w(x, t) = u(x, t)|_{x \geq 0},
\]
similar to the case of the Dirichlet problem. One can show that this \( w(x, t) \) solves the IBVP (9), and use the expression for the heat kernel, as well as the definition (10), to write the solution formula as follows
\[
w(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left[ e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] \phi(y) dy.
\]

Notice that the formulas (6) and (13) differ only by the sign between the two exponential terms inside the integral.

In terms of heat conduction, the Neumann condition in (9) means that there is no heat exchange between the rod and the environment (recall that the heat flux is proportional to the spatial derivative of the temperature). The physical interpretation of formula (13) is that the integrand is the contribution of \( \phi(y) \) plus an additional contribution, which comes from the lack of heat transfer to the points of the rod with negative coordinates.
Example 12.2. Solve the IBVP (9) with the initial data \( \phi(x) \equiv 1 \).

Using the formula (13), we can write the solution as

\[
\begin{align*}
    w(x, t) &= \frac{1}{\sqrt{4\pi k t}} \int_{0}^{\infty} \left[ e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right] dy \\
    &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2} dq,
\end{align*}
\]

where we made the changes of variables

\[
p = \frac{y - x}{\sqrt{4kt}}, \quad \text{and} \quad q = \frac{y + x}{\sqrt{4kt}}.
\]

Using the same idea as in the previous example, we can write the solution in terms of the \( \text{Erf} \) function as follows

\[
\begin{align*}
    w(x, t) &= \left[ \frac{1}{2} + \frac{1}{2} \text{Erf} \left( \frac{x}{\sqrt{4kt}} \right) \right] + \left[ \frac{1}{2} - \frac{1}{2} \text{Erf} \left( \frac{x}{\sqrt{4kt}} \right) \right] \\
    &\equiv 1.
\end{align*}
\]

So the solution is identically 1, which is clear if one thinks in terms of heat conduction. Indeed, problem (9) describes the temperature dynamics with identically 1 initial temperature, and no heat loss at the endpoint. Obviously there is no heat transfer between points of equal temperature, so the temperatures remain steady along the entire rod. \( \square \).

12.2 Conclusion

We derived the solution to the heat equation on the half-line by reducing the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In the case of zero Dirichlet boundary condition the odd extension of the initial data automatically guarantees that the solution will satisfy the boundary condition. While for the case of zero Neumann boundary condition the appropriate choice is the even extension. This reflection method relies on the fact that the solution to the heat equation on the whole line with odd initial data is odd, while the solution with even initial data is even.
We reduce the Dirichlet problem \((14)\) to the whole line by the reflection method. The idea is again to extend which is exactly d’Alembert’s formula.

Consider the IVP on the whole line with the extended initial data

\[
\phi(x) \quad \text{for } x > 0, \\
0 \quad \text{for } x = 0, \\
-\phi(-x) \quad \text{for } x < 0.
\]

For the vibrating string, the boundary condition of \((14)\) means that the end of the string at \(x = 0\) is held fixed. We reduce the Dirichlet problem \((14)\) to the whole line by the reflection method. The idea is again to extend the initial data, in this case \(\phi, \psi,\) to the whole line, so that the boundary condition is automatically satisfied for the solutions of the IVP on the whole line with the extended initial data. Since the boundary condition is in the Dirichlet form, one must take the odd extensions

\[
\phi_{\text{odd}}(x) = \begin{cases} 
\phi(x) & \text{for } x > 0, \\
0 & \text{for } x = 0, \\
-\phi(-x) & \text{for } x < 0.
\end{cases}
\]

\[
\psi_{\text{odd}}(x) = \begin{cases} 
\psi(x) & \text{for } x > 0, \\
0 & \text{for } x = 0, \\
-\psi(-x) & \text{for } x < 0.
\end{cases}
\]

Consider the IVP on the whole line with the extended initial data

\[
\begin{align*}
{u_{tt} - c^2 u_{xx} = 0,} & \quad -\infty < x < \infty, 0 < t < \infty, \\
u(x, 0) = \phi_{\text{odd}}(x), u_t(x, 0) = \psi_{\text{odd}}(x).
\end{align*}
\]

Since the initial data of the above IVP are odd, we know from a homework problem that the solution of the IVP, \(u(x, t)\), will also be odd in the \(x\) variable, and hence \(u(0, t) = 0\) for all \(t > 0\). Then defining the restriction of \(u(x, t)\) to the positive half-line \(x \geq 0\),

\[
v(x, t) = u(x, t)\Big|_{x \geq 0},
\]

we automatically have that \(v(0, t) = u(0, t) = 0\). So the boundary condition of the Dirichlet problem \((14)\) is satisfied for \(v\). Obviously the initial conditions are satisfied as well, since the restrictions of \(\phi_{\text{odd}}(x)\) and \(\psi_{\text{odd}}(x)\) to the positive half-line are \(\phi(x)\) and \(\psi(x)\) respectively. Finally, \(v(x, t)\) solves the wave equation for \(x > 0\), since \(u(x, t)\) satisfies the wave equation for all \(x \in \mathbb{R}\), and in particular for \(x > 0\). Thus, \(v(x, t)\) defined by \((17)\) is a solution of the Dirichlet problem \((14)\). It is clear that the solution must be unique, since the odd extension of the solution will solve IVP \((16)\), and therefore must be unique.

Using d’Alembert’s formula for the solution of \((16)\), and taking the restriction \((17)\), we have that for \(x \geq 0\),

\[
v(x, t) = \frac{1}{2} \left[ \phi_{\text{odd}}(x + ct) + \phi_{\text{odd}}(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{odd}}(s) \, ds.
\]

Notice that if \(x \geq 0\) and \(t > 0\), then \(x + ct > 0\), and \(\phi_{\text{odd}}(x + ct) = \phi(x + ct)\). If in addition \(x - ct > 0\), then \(\phi_{\text{odd}}(x - ct) = \phi(x - ct)\), and over the interval \(s \in [x - ct, x + ct]\), \(\psi_{\text{odd}}(s) = \psi(s)\). Thus, for \(x > ct\), we have

\[
v(x, t) = \frac{1}{2} \left[ \phi(x + ct) + \phi(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds,
\]

which is exactly d’Alembert’s formula.

For \(0 < x < ct\), the argument \(x - ct < 0\), and using \((15)\) we can rewrite the solution \((18)\) as

\[
v(x, t) = \frac{1}{2} \left[ \phi(x + ct) - \phi(ct - x) \right] + \frac{1}{2c} \left[ \int_{x-ct}^{0} -\psi(-s) \, ds + \int_{0}^{x+ct} \psi(s) \, ds \right].
\]
Making the change of variables $s \mapsto -s$ in the first integral on the right, we get
\[
 v(x, t) = \frac{1}{2} \left[ \phi(x + ct) - \phi(ct - x) \right] + \frac{1}{2c} \left[ \int_{ct-x}^{0} \psi(s) \, ds + \int_{0}^{x+ct} \psi(s) \, ds \right] 
 = \frac{1}{2} \left[ \phi(x + ct) - \phi(ct - x) \right] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(s) \, ds. 
\]

One could also use the fact that the integral of the odd function $\psi_{\text{odd}}(s)$ over the symmetric interval $[x - ct, ct - x]$ is zero, thus $\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s) \, ds = \int_{ct-x}^{x+ct} \psi(s) \, ds$.

The two different cases giving different expressions are illustrated in Figures 1 and 2 below. Notice how one of the characteristics from a point with $x_0 < ct_0$ gets reflected from the “wall” at $x = 0$ in Figure 2.

![Figure 1: The case with $x_0 > ct_0$.](image1)

![Figure 2: The case with $x_0 < ct_0$.](image2)

Combining the two expressions for $v(t, x)$ over the two regions, we arrive at the solution
\[
 v(x, t) = \begin{cases} 
 \frac{1}{2} \left[ \phi(x + ct) + \phi(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds, & \text{for } x > ct \\
 \frac{1}{2} \left[ \phi(x + ct) - \phi(ct - x) \right] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(s) \, ds, & \text{for } 0 < x < ct.
\end{cases} 
\]  

(20)

The minus sign in front of $\phi(ct - x)$ in the second expression above, as well as the reduction of the integral of $\psi$ to the smaller interval are due to the cancellation stemming from the reflected wave. The next example illustrates this phenomenon.

**Example 13.1.** Solve the Dirichlet problem (14) with the following initial data
\[
 \phi(x) = \begin{cases} 
 h & \text{for } a < x < 2a, \\
 0 & \text{otherwise},
\end{cases} \quad \text{and } \psi(x) \equiv 0.
\]

The initial data defines a box-like displacement over the interval $(a, 2a)$ and zero initial velocity for the string. It is clear that for some small time the wave will propagate just like in the case of the IVP on the whole line, i.e. without “seeing” the boundary. This is due to the finite speed of propagation property of the wave equation, according to which it takes some time, specifically $a/c$ time for the initial displacement to reach the boundary. Thus, we expect that the box-like wave will break into two box-waves, each with half the height of the initial box-like displacement, which travel with speed $c$ in opposite directions. The box-wave traveling in the right direction will never hit the boundary at $x = 0$, and will continue traveling unaltered for all time. However, the left box-wave hits the wall (the fixed end of the vibrating string), and gets reflected in an odd fashion, that is the displacement gets the minus sign in the second expression of (20).

To find the values of the solution at any point $(x_0, t_0)$, we draw the backward characteristics from that point and trace the point back to the $x$ axis, where the initial data is defined. Since the initial data is nonzero only over the interval $(a, 2a)$, only those characteristics that hit the $x$ axis between the points $a$ and $2a$ will carry nonzero values forward in time. Notice also that if a characteristic hits the interval $(a, 2a)$ after being reflected from the wall $x = 0$, then the value it gets must be taken with a minus sign.

The different values determined by this method are illustrated in Figure 3 below. Notice how the left box-wave with height $h/2$ flips after hitting the boundary, and travels in the opposite direction with negative height $-h/2$.

One can then use the values from Figure 3 to draw the profile of the string at any time $t$. □
Figure 3: The values of $v(x,t)$ carried forward in time by the characteristics.

13.1 Neumann boundary condition

For the Neumann problem on the half-line,

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & 0 < x < \infty, 0 < t < \infty, \\
    w(x,0) = \phi(x), & w_t(x,0) = \psi(x), \quad x > 0, \\
    w_x(0,t) = 0, & t > 0, 
\end{cases}$$

we use the reflection method with even extensions to reduce the problem to an IVP on the whole line. Define the even extensions of the initial data

$$\phi_{\text{even}} = \begin{cases} \phi(x) & \text{for } x \geq 0, \\
\phi(-x) & \text{for } x \leq 0, 
\end{cases} \quad \psi_{\text{even}} = \begin{cases} \psi(x) & \text{for } x \geq 0, \\
\psi(-x) & \text{for } x \leq 0. 
\end{cases}$$

and consider the following IVP on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\
    u(x,0) = \phi_{\text{even}}(x), & u_t(x,0) = \psi_{\text{even}}(x). 
\end{cases}$$

Clearly, the solution $u(x,t)$ to the IVP (23) will be even in $x$, and since the derivative of an even function is odd, $u_x(x,t)$ will be odd in $x$, and hence $u_x(0,t) = 0$ for all $t > 0$. Similar to the case of the Dirichlet problem, the restriction

$$w(x,t) = u(x,t) \bigg|_{x \geq 0}$$

will be the unique solution of the Neumann problem (21).

Using d’Alambert’s formula for the solution $u(x,t)$ of (23), and taking the restriction to $x \geq 0$, we get

$$w(x,t) = \frac{1}{2} \left[ \phi_{\text{even}}(x + ct) + \phi_{\text{even}}(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{even}}(s) \, ds. \quad (24)$$

One again needs to consider the two cases $x > ct$ and $0 < x < ct$ separately. Notice that with the even extensions we will get additions, rather than cancellations. Using the definitions (22), the solution (24) can be written as

$$w(x,t) = \begin{cases} \frac{1}{2} \left[ \phi(x + ct) + \phi(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) \, ds, & \text{for } x > ct \\
\frac{1}{2} \left[ \phi(x + ct) + \phi(ct - x) \right] + \frac{1}{2c} \left[ \int_{0}^{ct-x} \psi(s) \, ds + \int_{0}^{x+ct} \psi(s) \, ds \right], & \text{for } 0 < x < ct. 
\end{cases}$$

The Neumann boundary condition corresponds to a vibrating string with a free end at $x = 0$, since the string tension, which is proportional to the derivative $v_x(x,t)$, vanishes at $x = 0$. In this case the reflected wave adds to the original wave, rather than canceling it.
Example 13.2. Solve the Neumann problem (21) with the following initial data
\[
\phi(x) = \begin{cases} 
    h & \text{for } a < x < 2a, \\
    0 & \text{otherwise,}
\end{cases} 
\quad \psi(x) \equiv 0.
\]

The initial data is exactly the same as in the previous example for the Dirichlet problem. The only difference from the Dirichlet case is that the free end reflects the wave with a plus sign. The values carried forward in time by the characteristics are determined in the same way as before. Figure 4 illustrates this method. Notice that the reflected wave has the same (positive) height \( h/2 \) as the wave right before the reflection.

One can again use the values from Figure 4 to draw the profile of the string at any time \( t \).

13.2 Conclusion

We derived the solution to the wave equation on the half-line in much the same way as was done for the heat equation. That is, we reduced the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data. In this case the characteristics nicely illustrate the reflection phenomenon. We saw that the characteristics that hit the initial data after reflection from the boundary wall \( x = 0 \) carry the values of the initial data with a minus sign in the case of the Dirichlet boundary conditions, and with a plus sign in the case of the Neumann boundary conditions. This corresponds to our intuition of reflected waves from a fixed end, and free end respectively.
14 Waves on the finite interval

In the last lecture we used the reflection method to solve the boundary value problem for the wave equation on the half-line. We would like to apply the same method to the boundary value problems on the finite interval, which correspond to the physically realistic case of a finite string. Consider the Dirichlet wave problem on the finite line

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
u_{tt} - c^2 v_{xx} = 0, & 0 < x < l, 0 < t < \infty, \\
v(x, 0) = \phi(x), & v_t(x, 0) = \psi(x), & x > 0, \\
v(0, t) = v(l, t) = 0, & t > 0.
\end{array}
\right.
\end{align*}
\tag{25}
\]

The homogeneous Dirichlet conditions correspond to the vibrating string having fixed ends, as is the case for musical instruments. Using our intuition from the half-line problems, where the wave reflects from the fixed end, we can imagine that in the case of the finite interval the wave bounces back and forth infinitely many times between the endpoints. In spite of this, we can still use the reflection method to find the value of the solution to problem (25) at any point \((x, t)\).

Recall that the idea of the reflection method is to extend the initial data to the whole line in such a way, that the boundary conditions are automatically satisfied. For the homogeneous Dirichlet data the appropriate choice is the odd extension. In this case, we need to extend the initial data \(\phi, \psi\), which are defined only on the interval \(0 < x < l\), in such a way that the resulting extensions are odd with respect to both \(x = 0\), and \(x = l\). That is, the extensions must satisfy

\[f(-x) = -f(x) \quad \text{and} \quad f(l - x) = -f(l + x).\]  
\tag{26}

Notice that for such a function \(f(0) = -f(0)\) from the first condition, and \(f(l) = -f(l)\) from the second condition, hence, \(f(0) = f(l) = 0\). Subsequently, the solution to the IVP with such data will be odd with respect to both \(x = 0\) and \(x = l\), and the boundary conditions will be automatically satisfied. Notice that the conditions (26) imply that functions that are odd with respect to both \(x = 0\) and \(x = l\) satisfy \(f(2l + x) = -f(-x) = f(x)\), which means that such functions must be \(2l\)-periodic. Using this we can define the extensions of the initial data \(\phi, \psi\) as

\[
\phi_{\text{ext}}(x) = \begin{cases} 
\phi(x) & \text{for } 0 < x < l, \\
-\phi(-x) & \text{for } -l < x < 0, \\
etended to be } 2l& \text{periodic,}
\end{cases}
\quad \psi_{\text{ext}}(x) = \begin{cases} 
\psi(x) & \text{for } 0 < x < l, \\
-\psi(-x) & \text{for } -l < x < 0, \\
etended to be } 2l& \text{periodic.}
\end{cases}
\tag{27}
\]

Now, consider the IVP on the whole line with the extended initial data

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, 0 < t < \infty, \\
u(x, 0) = \phi_{\text{ext}}(x), u_t(x, 0) = \psi_{\text{ext}}(x).
\end{array}
\right.
\end{align*}
\]

For the solution of this IVP we automatically have \(u(0, t) = u(l, t) = 0\), and the restriction

\[v(x, t) = u(x, t)\big|_{0 \leq x \leq l},\]

will solve the boundary value problem (25). By d’Alambert’s formula, the solution will be given as

\[v(x, t) = \frac{1}{2} [\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{ext}}(s) \, ds\]  
\tag{28}

for \(0 < x < l\). Although formula (28) contains all the information about our solution, we would like to have an expression in terms of the original initial data, so that the values of the solution can be directly computed using the given functions \(\phi(x)\) and \(\psi(x)\). For this, we need to “bring” the points \(x - ct\) and \(x + ct\) into the interval \((0, l)\) using the periodicity and oddity of the extended data. To illustrate how this is done, let us fix a point \((x, t)\) and try to find the value of the solution at this point by tracing it back in time along the characteristics to the initial data. The sketch of the backwards characteristics from this point appears in Figure 5 above.
In general, the points $x + ct$ and $x - ct$ will end up either in the interval $(0, l)$ or $(-l, 0)$ after finitely many translations by the period $2l$. If the point ends up in $(0, l)$ (even number of reflections), then the value of the initial data picked up by the reflected characteristic will be taken with a positive sign. If, however, the point ends up in the interval $(-l, 0)$ (odd number of reflections), then we need to reflect this point with respect to $x = 0$, and the corresponding value of the initial data will be taken with a negative sign.

From Figure 5 we see that $x + ct$ goes into the interval $(0, l)$ (2 reflections) after translating it to the left by one period $2l$, but the point $x - ct$ goes into the interval $(-l, 0)$ (3 reflections) after a right translation by $2l$, so we need to reflect the resulting point $x - ct + 2l$ to arrive at the point $ct - x - 2l$ in the interval $(0, l)$. The solution will then be

$$u(x, t) = \frac{1}{2} [\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c} \int_{x+ct}^{x+ct-2l} \psi_{\text{ext}}(s) \, ds.$$  

For the integral term, we can break it into two integrals as follows

$$\frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{ext}}(s) \, ds = \frac{1}{2c} \int_{x-ct}^{ct-x} \psi_{\text{ext}}(s) \, ds + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi_{\text{ext}}(s) \, ds.$$  

Notice that from the oddity of $\psi_{\text{ext}}$, the integral over the interval $[x - ct, ct - x]$ will be zero, while by periodicity, we can bring the interval $[ct - x, x + ct]$ into the interval $(0, l)$ by subtracting one period $2l$. Thus, the solution can be written as

$$u(x, t) = \frac{1}{2} [\phi(x + ct - 2l) - \phi(ct - x - 2l)] + \frac{1}{2c} \int_{ct-x-2l}^{x+ct-2l} \psi(s) \, ds. \quad (29)$$  

Clearly, the derivation of the above expression for the solution depends on the chosen point, which in turn determines how many reflections the backward characteristics undergo before arriving at the $x$ axis. Hence, the solution will be given by different expressions, depending on the region from which the point is taken. These different regions are depicted in Figure 6, where the labels $(m, n)$ show how many times each of the two backward characteristics gets reflected before reaching the $x$ axis. Expression (29) will be valid for all the points in the region $(3, 2)$.

The method used to arrive at the expression (29) can be used to find the value of the solution at any point $(x, t)$, although it is quite impractical to derive the expression for each of the regions depicted in Figure 6. Furthermore, it does not generalize to higher dimensions, nor does it apply to the heat equation (no characteristics to trace back). Later on we will study another method, called separation of variables, which allows for a more general way of approaching boundary value problems on finite intervals.
Figure 6: Regions of $(x, t) \in (0, l) \times (0, \infty)$ with the different number of reflections.

Example 14.1. Consider the Dirichlet wave problem on the finite interval

\[
\begin{align*}
    &u_{tt} - u_{xx} = 0, \quad \text{for } 0 < x < 1, \\
    &u(x, 0) = x(1 - x), \quad u_t(x, 0) = x^2, \\
    &u(0, t) = u(1, t) = 0.
\end{align*}
\]

Find the value of the solution at the point $(\frac{3}{4}, \frac{5}{2})$.

Notice that in this problem $c = 1$, and $l = 1$, so the period of the extended data will be $2l = 2$. The sketch of the backward characteristics from the point $(x, t) = (\frac{3}{4}, \frac{5}{2})$ appears in the figure below.

Figure 7: The backwards characteristics from the point $(\frac{3}{4}, \frac{5}{2})$.

The characteristics intersect the $x$ axis at the points

\[ x - t = \frac{3}{4} - \frac{5}{2} = -1\frac{3}{4} \quad \text{and} \quad x + t = \frac{3}{4} + \frac{5}{2} = 3\frac{1}{4}. \]

The point $-1\frac{3}{4}$ goes to the point $\frac{1}{4}$ after a right translation by one period, while the point $3\frac{1}{4}$ goes to the point $1\frac{1}{4}$ after a left translation by one period. After a reflection with respect to $x = 1$, this point will end up at $\frac{3}{4}$; thus, the value of the initial data must be taken with a negative sign at this point. Also, the integral over the
interval \([\frac{3}{4}, 1]\) of \(\psi_{\text{ext}}\) will be zero due to its oddity with respect to \(x = 1\). The value of the solution is then

\[
u(\frac{3}{4}, \frac{5}{2}) = \frac{1}{2}[-\phi(\frac{3}{4}) + \phi(\frac{1}{4})] + \frac{1}{2} \int_{\frac{1}{4}}^{\frac{3}{4}} \psi(s) \, ds = \frac{1}{2} \left[ -\frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} \right] + \frac{1}{2} \int_{\frac{1}{4}}^{\frac{3}{4}} x^2 \, dx
\]

\[
= \frac{x^3}{6} \bigg|_{\frac{1}{4}}^{\frac{3}{4}} = \frac{1}{6} \left( \frac{27}{64} - \frac{1}{64} \right) = \frac{13}{192}.
\]

\[\square\]

### 14.1 The parallelogram rule

Recall from a homework problem, that for the vertices of a characteristic parallelogram \(A, B, C\) and \(D\) as for example in Figure 8, the values of the solution of the wave equation are related as follows

\[
u(A) + \nu(C) = \nu(B) + \nu(D).
\]

Hence, we can find the value at the vertex \(A\) from the values at the three other vertices.

\[
u(A) = \nu(B) + \nu(D) - \nu(C).
\]

Notice that the values at the vertices \(B\) and \(C\) in Figure 8 can be found from the expression of the solution for the region \((0, 0)\), while the value at \(D\) comes from the boundary data. Thus we reduced finding the value at a point in the region \((1, 0)\) to finding values in the region \((0, 0)\). One can always follow this procedure to evaluate the solution in the regions \((m+1, n)\) and \((m, n+1)\) via the values in the region \((m, n)\), provided the boundary condition is in the Dirichlet form.

![Figure 8: The parallelogram rule.](image)

### 14.2 Conclusion

We applied the reflection method to derive expressions for the solution to the Dirichlet wave problem on the finite interval. However, the method yields infinitely many expressions for different regions in \((x, t) \in (0, l) \times (0, \infty)\), depending on the number of times the backward characteristics from a point get reflected before reaching the \(x\) axis, where the initial data is defined. This makes the method impractical in applications, and is not generalizable to higher dimensions and other PDEs. An alternative method (separation of variables) of solving boundary value problems on the finite interval will be described later in the course.