1. (#2.3.5 in Strauss) The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.

(a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$.

(b) Where precisely does the proof of the maximum principle break down for this equation?

2. (#2.4.9 in Strauss) Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that $u_{xxx}$ satisfies the diffusion equation with zero initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x, t) = A(t)x^2 + B(t)x + C(t)$. Finally, it’s easy to solve for $A, B, C$ by plugging into the original problem.

3. (#2.4.4 in Strauss) Solve the heat equation if $\phi(x) = e^{-x}$ for $x > 0$, and $\phi(x) = 0$ for $x < 0$.

4. (#2.4.11 (a) in Strauss) Consider the heat equation on the whole line with the usual initial condition $u(x, 0) = \phi(x)$. If $\phi(x)$ is an odd function, show that the solution $u(x, t)$ is also an odd function of $x$. (Hint: Consider $u(-x, t) + u(x, t)$ and use the uniqueness.)

5. (#2.4.15 in Strauss) Prove the uniqueness of the heat problem with Neumann boundary conditions:

$$\begin{cases}
  u_t - ku_{xx} = f(x, t) & \text{for } 0 < x < l, t > 0, \\
  u(x, 0) = \phi(x), \\
  u_x(0, t) = g(t), \quad u_x(l, t) = h(t),
\end{cases}$$

by the energy method.

6. (#2.4.16 in Strauss) Solve the initial value problem for the diffusion equation with constant dissipation:

$$\begin{cases}
  u_t - ku_{xx} + bu = 0 & \text{for } -\infty < x < \infty, \\
  u(x, 0) = \phi(x),
\end{cases}$$

where $b > 0$ is a constant. (Hint: Make the change of variables $u(x, t) = e^{-bt}v(x, t)$.)

7. (#2.5.4 in Strauss) Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x, t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2 c^2 / 4kt} u(x, s) \, ds.$$ 

(a) Show that $v(x, t)$ solves the diffusion equation!

(b) Show that $\lim_{t \to 0} v(x, t) = u(x, 0)$.

(Hint: (a) Write the formula as $v(x, t) = \int_{-\infty}^{\infty} H(s, t)u(x, s) \, ds$, where $H(x, t)$ solves the diffusion equation with constant $k/c^2$ for $t > 0$. Then differentiate $v(x, t)$, assuming that you can freely differentiate inside the integral. (b) Use the fact that $H(s, t)$ is essentially the diffusion (heat) kernel with the spatial variable $s$. You can use the fact that the diffusion kernel has the Dirac delta function as its initial data.)