Inverse of a matrix

If $A$ is an $n \times n$ matrix, an inverse of $A$ is an $n \times n$ matrix $A'$, with the property that

$$AA' = A'A = I_n \quad \text{and} \quad A'A = I_n.$$

If such an $A'$ exists, then $A$ is called invertible.

Some of the properties of invertible matrices are stated in the following theorems.

**Theorem**: If $A$ is invertible, then its inverse is unique. This unique inverse is denoted $A^{-1}$.

**Theorem**: If $A$ is an invertible $n \times n$ matrix, then the system of linear equations given by $Ax = b$ has the unique solution $x = A^{-1}b$ for any $b$ in $\mathbb{R}^n$.

Although it’s hard to find the inverse of a general $n \times n$ matrix, it is fairly simple when $n = 2$.

**Theorem**: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A$ is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ 

**Properties**:

- $(A^{-1})^{-1} = A$
- $(cA)^{-1} = cA^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A^n)^{-1} = (A^{-1})^n$

An **elementary matrix** is any matrix that can be obtained by performing an elementary row operation on an identity matrix. Since there are three different types of elementary row operations, we will distinguish corresponding three types of elementary matrices.

We saw earlier that the elementary row operations are invertible, so each elementary matrix is invertible and its inverse is an elementary matrix of the same type. Elementary matrices offer a useful tool, which allows to represent the action of elementary row operations via matrix multiplication.

**Theorem**: If $E$ is an elementary matrix obtained by performing an elementary row operation on $I_n$, then the result of the same operation on an $n \times r$ matrix $A$ is $EA$.

**Example**:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + 2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = E, \quad A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}.$$

$$EA =$$
**Fundamental Theorem:** Let $A$ be an $n \times n$ matrix. The following statements are equivalent.

(a) $A$ is invertible

(b) $Ax = b$ has a unique solution for every $b$ in $\mathbb{R}^n$.

(c) $Ax = 0$ has only the trivial solution.

(d) The reduced row echelon form of $A$ is $I_n$.

(e) $A$ is a product of elementary matrices.

One corollary from the fundamental theorem is the following result.

**Theorem:** Let $A$ be a square matrix. If $B$ is a square matrix such that $AB = I$ or $BA = I$, then $A$ is invertible and $B = A^{-1}$.

The fundamental theorem also offers a relatively straightforward method of obtaining the inverse of a matrix via Gauss-Jordan elimination.

**Theorem:** Let $A$ be a square matrix. If a sequence of elementary row operations reduces $A$ to $I$, then the same sequence of elementary row operations transforms $I$ into $A^{-1}$.

So to find the inverse of $A$, form the block matrix $[A|I]$ and reduce the $A$ block to $I$ by Gauss Jordan elimination. The second block will then automatically transform into $A^{-1}$.

**Example:** Find the inverse of the following matrix by Gauss-Jordan elimination

\[
\begin{pmatrix}
1 & 2 & -1 \\
2 & 5 & -1 \\
1 & 2 & 0 \\
\end{pmatrix}
\]
Exercises:

1. Find the inverse of the matrix by Gauss-Jordan elimination
   \[
   \begin{bmatrix}
   1 & -2 & 1 \\
   -3 & 7 & -6 \\
   2 & -3 & 0 
   \end{bmatrix}
   \]

2. Determine whether the matrix is invertible, and find its inverse if it is.
   \[
   \begin{bmatrix}
   1 & 1 & -2 \\
   2 & 1 & -3 \\
   -3 & -1 & 4 
   \end{bmatrix}
   \]