Matrix algebra

The square matrix \( I = [\delta_{ij}]_{n \times n} \) is called an **identity matrix** if

\[
\delta_{ij} = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{for } i \neq j
\end{cases}
\]

That is, the identity matrix has the form

\[
I = [\delta_{ij}]_{n \times n} = \begin{bmatrix} 
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}.
\]

If \( A = A_{m \times n} \) and \( B = B_{n \times r} \), then

\[
AI = A \quad \text{and} \quad IB = B.
\]

**Theorem:** Let \( A \) be an \( n \times m \) matrix, \( e_i \) be a \( 1 \times m \) standard unit vector, and \( e_j \) an \( n \times 1 \) standard unit vector. Then

(a) \( e_i A \) is the \( i \)th row of \( A \)

(b) \( Ae_j \) is the \( j \)th column of \( A \).

**Matrix column representation** of matrix multiplication: Let \( A_{m \times n} \) be an \( m \times n \) matrix, and \( B_{n \times r} \) be written in terms of column vectors \( B = [ b_1 \ b_2 \ \ldots \ b_r ]_{n \times r} \), then

\[
AB = A[ b_1 \ b_2 \ \ldots \ b_r ]_{n \times r} = [ Ab_1 \ Ab_2 \ \ldots \ Ab_r ]_{n \times r}.
\]

**Matrix row representation** of matrix multiplication: Let \( A_{m \times n} \) be an \( m \times n \) matrix written in terms of row vectors \( A = [ A_1 \ A_2 \ \ldots \ A_m ]_{m \times n} \) and \( B_{n \times r} \) be an \( n \times r \) matrix, then

\[
AB = [ A_1 ]_{m \times n} B = [ A_1 B ]_{m \times r}.
\]

**Column row representation** of matrix multiplication: Let \( A_{m \times n} = [ a_1 \ a_2 \ \ldots \ a_r ]_{m \times n} \), and \( B = [ B_1 \ B_2 \ \ldots \ B_n ]_{n \times r} \), then

\[
AB = [ a_1 \ a_2 \ \ldots \ a_r ]_{m \times n} [ B_1 \ B_2 \ \ldots \ B_n ]_{n \times r} = a_1 B_1 + a_2 B_2 + \cdots + a_r B_n.
\]
Matrix powers  

\[ A^k = A \cdot A \cdots A, \quad k \text{ times.} \]

Matrix powers have the following properties

(a) \( A^r A^s = A^{r+s} \).
(b) \( (A^r)^s = A^{rs} \).

Transpose of a matrix: Let \( A = [a_{ij}]_{m \times n} \) be an \( m \times n \) matrix, then its transpose, \( A^T \) is the \( n \times m \) matrix

\[ A^T = [a_{ji}]_{n \times m}. \]

A square matrix is **symmetric** if \( A^T = A \). This is equivalent to \( A_{ij} = A_{ji} \) for all \( i, j \).

Properties of matrix operations: \( A, B, C \) are matrices, \( c, d, k \) are scalars, \( O \) is a zero matrix, \( I \) is an identity matrix.

**Addition and scalar multiplication:**

(a) \( A + B = B + A \)  
(b) \( (A + B) + C = A + (B + C) \)  
(c) \( A + O = A \)  
(d) \( A + (-A) = O \)  
(e) \( c(A + B) = cA + cB \)  
(f) \( (c + d)A = cA + dA \)  
(g) \( c(dA) = (cd)A \)  
(h) \( 1A = A \)

Matrix multiplication:

(a) \( A(BC) = (AB)C \)  
(b) \( A(B + C) = AB + AC \)  
(c) \( (A + B)C = AC + BC \)  
(d) \( k(AB) = (kA)B = A(kB) \)  
(e) \( I_mA = A = AI_n, \text{ if } A = A_{m \times n} \)

Transpose of a matrix:

(a) \( (A^T)^T = A \)  
(b) \( (A + B)^T = A^T + B^T \)  
(c) \( (kA)^T = k(A^T) \)  
(d) \( (AB)^T = B^T A^T \)  
(e) \( (A^r)^T = (A^T)^r, r \geq 0 \)

Theorem:

(a) If \( A \) is a square matrix, then \( A + A^T \) is a symmetric matrix.

(b) For any matrix \( A \), matrices \( AA^T \) and \( A^T A \) are symmetric matrices.
Exercises:

1. Write the matrix $B$ as a linear combination of $A_1$ and $A_2$, where
   \[
   B = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}
   \]

2. Find a $2 \times 2$ non-zero matrix $A$, such that $A^2 = O$ (zero matrix).

3. The trace of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of the diagonal entries, that is
   \[
   \text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}
   \]

   Show that if $A$ and $B$ are $n \times n$ matrices, then
   \[
   \text{tr}(AB) = \text{tr}(BA)
   \]