Spanning sets and linear independence

A linear combination of vectors $v_1, v_2, \cdots, v_k$ is

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

for any choice of scalars $c_1, c_2, \ldots, c_k$.

**Theorem:** A system of linear equations with augmented matrix $[A|b]$ is consistent if and only if $b$ is a linear combination of columns of $A$.

The span of a set of vectors $S = \{v_1, v_1, \ldots, v_k\}$, denoted by $\text{span}(S)$ or $\text{span}(v_1, v_2, \ldots, v_k)$, is the set of all linear combinations of $v_1, v_1, \ldots, v_k$. If $\text{span}(S) = \mathbb{R}^n$, then $S$ is called a spanning set of $\mathbb{R}^n$.

A set of vectors $v_1, v_2, \ldots, v_k$ is called linearly dependent, if there are scalars $c_1, c_2, \ldots, c_k$, at least one of which is nonzero, such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0.$$

A set of vectors which is not linearly dependent is called linearly independent.

Below are some criteria for linear (in)dependence:

**Theorem:** Vectors $v_1, v_2, \ldots, v_k$ in $\mathbb{R}^n$ are linearly dependent if and only if at least one of them can be expressed as a linear combination of the others.

**Remark:** Any set of vectors containing 0 is linearly dependent.

**Theorem:** Let $v_1, v_2, \ldots, v_k$ be column vectors in $\mathbb{R}^n$ and let $A$ be the $n \times m$ matrix $[v_1v_2\ldots v_k]$ with these vectors as its columns. Then $v_1, v_2, \ldots, v_k$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A|0]$ has a nontrivial solution.

**Theorem:** Let $v_1, v_2, \ldots, v_k$ be row vectors in $\mathbb{R}^n$ and let $A$ be the $m \times n$ matrix $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$ with these vectors as its rows. Then $v_1, v_2, \ldots, v_k$ are linearly dependent if and only if $\text{rank}(A) < m$.

**Theorem:** Any set of $m$ vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$. 
Exercises:

1. Find the span of the vectors \[ \begin{bmatrix} 2 \\ -4 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \end{bmatrix} \].

2. Determine whether the vectors \( \mathbf{u} = (0, 1, 2), \mathbf{v} = (2, 1, 3) \) and \( \mathbf{w} = (2, 0, 1) \) are linearly dependent or independent.