Gaussian elimination

The augmented matrix of a system of linear equations has the form \([A|b]\), where \(A\) is the matrix of coefficients of the linear system.

Gaussian elimination is a method of reducing the linear system to an equivalent system which can be easily solved by back substitution. For this, we define:

A matrix is in row echelon form, if it has a staircase pattern, or more precisely:

1. Any rows that contain only zeros are at the bottom
2. In any nonzero row, the first nonzero entry (called the leading term of the row) is in a column to the left of any leading entries below it.

One can reduce any matrix to its row echelon form by performing elementary row operations.

Elementary row operations are:

1. Interchange two rows \((R_i \leftrightarrow R_j)\)
2. Multiply a row by a nonzero constant \((R_i = \lambda R_i)\)
3. Add a multiple of a row to another row \((R_i = R_i + \lambda R_j)\)

Notice that the elementary row operations are invertible, and we will call two matrices \(A\) and \(B\) row-equivalent if there are elementary row operations converting \(A\) to \(B\). Clearly two matrices are row equivalent, if and only if they have the same row echelon form (since one uses only invertible row operations to convert a matrix to a row echelon form).

Gaussian elimination

- Write the augmented matrix of a system of equations
- Use elementary row operations to reduce it to row echelon form
- Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.
Exercises:

1. Reduce the following matrix to its row echelon form (keep track of elementary row operations)

\[
\begin{bmatrix}
1 & 3 & 4 & -1 \\
2 & -1 & 5 & 1 \\
3 & 2 & 0 & 4 \\
\end{bmatrix}
\]

2. Solve the following system of linear equations by Gaussian elimination

\[
\begin{align*}
2x - y + z &= 3 \\
x + y - z &= 0 \\
3x + y - 2z &= -1
\end{align*}
\]