Planes

A plane in $\mathbb{R}^3$ is uniquely determined by

- A point $P$ on the plane and a vector normal to the plane, $\mathbf{n}$
- A point $P$ on the plane and two vectors parallel to the plane, $\mathbf{u}, \mathbf{v}$.
- Three points on the plane, $P$, $Q$, and $R$.

Let $\mathbf{p}$ be the position vector of the point $P = (x_0, y_0, z_0)$, then for the point with position vector $\mathbf{x} = (x, y, z)$ to be on the plane, the vector $\mathbf{x} - \mathbf{p}$ must be on the plane as well. This can be written as

\[
(x - p) \cdot \mathbf{n} = 0 \quad \text{(normal form, only in \(\mathbb{R}^3\))}
\]
\[
(x - p) = s \mathbf{u} + t \mathbf{v} \quad \text{(vector form, works in any \(\mathbb{R}^n, n \geq 3\))}
\]
\[
(x - p) = s(\mathbf{q} - \mathbf{p}) + t(\mathbf{r} - \mathbf{p})
\]

The third equation is a particular case of the vector form, since the vectors $\mathbf{q} - \mathbf{p}$ and $\mathbf{r} - \mathbf{p}$ are parallel to the plane. The vector form of the equation works in any higher dimensions. If we write the vector form in terms of components, this will give the parametric equations of the plane.

\[
\begin{align*}
x &= x_0 + su_1 + tv_1 \\
y &= y_0 + su_2 + tv_2 \\
z &= z_0 + su_3 + tv_3
\end{align*}
\]

The normal form of the equation is usually written as ($\mathbf{n} = (a, b, c)$)

\[
\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \quad \text{or} \quad ax + by + cz = d.
\]
Exercises:

1. Find the vector and parametric equations of the plane passing through the point \( P = (4, -1, 3) \) and parallel to the vectors \( \mathbf{u} = (1, 1, 0) \) and \( \mathbf{v} = (-1, 1, 1) \).

2. Find the equation of the set of all points that are equidistant from the points \( P = (1, 0, -2) \) and \( Q = (5, 2, 4) \).