Eigenvalues and eigenvectors

An eigenvalue of a square matrix $A$ is any number $\lambda$, such that the homogeneous linear system of equations $(A - \lambda I)x = 0$ has a nonzero solution (equivalently, $Ax = \lambda x$ has a nontrivial solution). Such nonzero solutions are called eigenvectors of $A$. As this is equivalent to the matrix $A - \lambda I$ being noninvertible, it is clear that the eigenvalues of a square matrix $A$ are precisely the solutions of the equation

$$\det(A - \lambda I) = 0.$$ 

The polynomial $\det(A - \lambda I)$ is called the characteristic polynomial of $A$, while the equation $\det(A - \lambda I) = 0$ is called the characteristic equation of $A$.

Here is the procedure to find eigenvalues and eigenvectors of an $n \times n$ matrix $A$:

(a) Compute the characteristic polynomial $\det(A - \lambda I)$.

(b) Solve the characteristic equation of $A$, $\det(A - \lambda I) = 0$, to find the eigenvalues of $A$.

(c) For each eigenvalue $\lambda$ find the nullspace of the matrix $A - \lambda I$. This is the eigenspace $E_\lambda$, the nonzero vectors of which are the eigenvectors of $A$ corresponding to $\lambda$.

(d) Find a basis for each eigenspace.

Algebraic multiplicity of an eigenvalue $\lambda$ is its multiplicity as a root of the characteristic polynomial. Geometric multiplicity of $\lambda$ is the dimension of the corresponding eigenspace $E_\lambda$.

Example: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -2 & 0 \end{bmatrix}$.
The following are simple corollaries from the fact that eigenvalues solve characteristic polynomials.

**Theorem:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

**Theorem:** A square matrix $A$ is invertible if and only if 0 is not an eigenvalue of $A$.

In the light of the last theorem, we can extend the Fundamental theorem for invertible matrices as follows.

**The Fundamental Theorem for Invertible Matrices:** Let $A$ be an $n \times n$ matrix. The following are equivalent.

(a) $A$ is invertible

(b) $Ax = b$ has a unique solution for every $b$ in $\mathbb{R}^n$.

(c) $Ax = 0$ has only the trivial solution.

(d) The reduced row echelon form of $A$ is $I_n$.

(e) $A$ is a product of elementary matrices.

(f) $\text{rank}(A) = n$.

(g) $\text{nullity}(A) = 0$.

(h) The column vectors of $A$ are linearly independent.

(i) The column vectors of $A$ span $\mathbb{R}^n$.

(j) The column vectors of $A$ form a basis for $\mathbb{R}^n$.

(k) The row vectors of $A$ are linearly independent.

(l) The row vectors of $A$ span $\mathbb{R}^n$.

(m) The row vectors of $A$ form a basis for $\mathbb{R}^n$.

(n) $\det A \neq 0$.

(o) 0 is not an eigenvalue of $A$.

**Theorem:** Let $A$ be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ and corresponding eigenvalues $v_1, v_2, \ldots, v_m$. Then $v_1, v_2, \ldots, v_m$ are linearly independent.
Exercises:

1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 2 & -2 & 2 \\ 2 & 0 & 0 \end{bmatrix}$, and check that eigenvectors corresponding to distinct eigenvalues form a linearly independent set.