Determinants, part 2

The following theorem illustrates how determinants and matrix operations are related.

**Theorem:** Let $A$ and $B$ be $n \times n$ matrices, and $k$ be a scalar.

1. $\det(kA) = k^n \det A$.
2. $\det(AB) = (\det A) \cdot (\det B)$.
3. If $A$ is invertible, then $\det(A^{-1}) = \frac{1}{\det A}$.
4. $\det A = \det(A^T)$.

Determinants can also be used to solve systems of linear equations and to find the inverse of a matrix. We first make the following notation:

If $A$ is an $n \times n$ matrix, and $b$ is a vector in $\mathbb{R}^n$, then $A_i(b)$ stands for the matrix in which the $i^{th}$ column of $A$ is replace by $b$. That is, if $A = [a_1 \ a_2 \ \ldots \ a_n]$, then $A_i(b) = [a_1 \ \ldots \ a_{i-1} \ b \ a_{i+1} \ \ldots \ a_n]$.

**Cramer’s Rule:** Let $A$ be an $n \times n$ invertible matrix, $b$ a vector in $\mathbb{R}^n$. Then the unique solution of the system $Ax = b$ is given by

$$x_i = \frac{\det(A_i(b))}{\det A}, \quad \text{for } i = 1, 2, \ldots, n.$$

**Example:**

Use Cramer’s rule to solve the following system of equations

$$
\begin{align*}
x_1 - x_2 &= 2 \\
-x_1 + 3x_2 &= 4.
\end{align*}
$$

The inverse of a matrix $A$ can be found by solving the matrix equation $AX = I$, which can be thought of as $n$ systems of equations, $Ax_j = e_j, \ j = 1, 2, \ldots, n$, where $x_j$ is the $j^{th}$ column of $X$, that is, $X = [x_1 \ x_2 \ \ldots \ x_n]$. Then it’s not hard to see that by Cramer’s rule the $i^{th}$ component of $x_j$ is given by

$$x_{ij} = \frac{\det A_i(e_j)}{\det A} = \frac{(-1)^{j+i} \det A_{ji}}{\det A} = \frac{C_{ji}}{\det A}.$$

If we now make the notation for the adjoint matrix of $A$,

$$\text{adj}A = [C_{ji}] = [C_{ij}]^T,$$

then by the above calculation we will have that

$$A^{-1} = \frac{1}{\det A} \text{adj}A.$$
Exercises:

1. Find the inverse of the following matrix

\[ A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 2 & 1 \\ -2 & 3 & 5 \end{bmatrix} \]