**Basis, Dimension, Rank**

A **basis** for a subspace $S$ of $\mathbb{R}^n$ is a set of vectors in $S$ that

1. span $S$
2. are linearly independent

An example of a basis is $\{e_1, e_2, \ldots, e_n\}$, which spans $\mathbb{R}^n$ and is linearly independent. This is called the **standard basis** of $\mathbb{R}^n$.

**Example:**
Find a basis for $\text{row}(A)$, $\text{col}(A)$ and $\text{null}(A)$, where

$$A = \begin{bmatrix}
1 & -2 & 1 & 4 \\
-1 & 2 & 1 & 2 \\
2 & -4 & 0 & 2
\end{bmatrix}.$$

In general, to find a basis for $\text{row}(A)$, $\text{col}(A)$ and $\text{null}(A)$, do the following:

1. Find the reduced row echelon form $R$ of $A$.
2. The nonzero row vectors of $R$ form a basis for $\text{row}(A)$.
3. The column vectors of $A$ corresponding to columns of $R$ with leading 1’s (the pivot columns) form a basis for $\text{col}(A)$.
4. Solve for the leading variables of $Rx = 0$ in terms of the free variables. Write the solution $x$ as a column vector expanded into a linear combination with the free variables being the coefficients. The vectors in terms of which the linear combination is written form a basis for $\text{null}(A)$.

Although the same subspace can have many different bases, the number of vectors in each must be the same.

**Theorem:** Let $S$ be a subspace of $\mathbb{R}^n$. Then the number of vectors in any basis of $S$ is the same and is called the **dimension** of $S$.

For the row and column space of a matrix the following property holds.

**Theorem:** The row and column space of a matrix $A$ have the same dimension.

The **rank** of a matrix $A$, denoted $\text{rank}(A)$, is the dimension of its row and column spaces. The **nullity** of a matrix $A$, denoted $\text{nullity}(A)$, is the dimension of its null space.

It is easy to see that $\text{rank}(A^T) = \text{rank}(A)$. The rank and the nullity of a matrix have the following relation.

**The Rank Theorem:** If $A$ is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$
The fundamental theorem for invertible matrices can be now extended using some of these new notions.

**The Fundamental Theorem for Invertible Matrices:** Let $A$ be an $n \times n$ matrix. The following are equivalent.

(a) $A$ is invertible

(b) $Ax = b$ has a unique solution for every $b$ in $\mathbb{R}^n$.

(c) $Ax = 0$ has only the trivial solution.

(d) The reduced row echelon form of $A$ is $I_n$.

(e) $A$ is a product of elementary matrices.

(f) rank($A$) = $n$.

(g) nullity($A$) = 0.

(h) The column vectors of $A$ are linearly independent.

(i) The column vectors of $A$ span $\mathbb{R}^n$.

(j) The column vectors of $A$ form a basis for $\mathbb{R}^n$.

(k) The row vectors of $A$ are linearly independent.

(l) The row vectors of $A$ span $\mathbb{R}^n$.

(m) The row vectors of $A$ form a basis for $\mathbb{R}^n$.

An application of the Rank Theorem and the Fundamental theorem gives the following.

**Theorem:** Let $A$ be an $m \times n$ matrix. Then:

- (a) $\text{rank}(A^T A) = \text{rank}(A)$.

- (b) The $n \times n$ matrix $A^T A$ is invertible if and only if $\text{rank}(A) = n$.

**Exercises:**

1. Determine whether the following vectors form a basis for $\mathbb{R}^3$.

$$
\begin{bmatrix}
1 \\
-1 \\
2
\end{bmatrix},
\begin{bmatrix}
-2 \\
1 \\
-5
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
$$