Basis, Dimension, Rank

A **basis** for a subspace S of $\mathbb{R}^n$ is a set of vectors in S that

1. span S
2. are linearly independent

An example of a basis is $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, which spans $\mathbb{R}^n$ and is linearly independent. This is called the **standard basis** of $\mathbb{R}^n$.

Example:
Find a basis for row(A), col(A) and null(A), where

$$A = \begin{bmatrix} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & 2 \\ 2 & -4 & 0 & 2 \end{bmatrix}.$$

In general, to find a basis for row(A), col(A) and null(A), do the following:

1. Find the reduced row echelon form R of A.
2. The nonzero row vectors of R form a basis for row(A).
3. The column vectors of A corresponding to columns of R with leading 1's (the pivot columns) form a basis for col(A).
4. Solve for the leading variables of $R\mathbf{x} = \mathbf{0}$ in terms of the free variables. Write the solution $\mathbf{x}$ as a column vector expanded into a linear combination with the free variables being the coefficients. The vectors in terms of which the linear combination is written form a basis for null(A).

Although the same subspace can have many different bases, the number of vectors in each must be the same.

Theorem: Let S be a subspace of $\mathbb{R}^n$. Then the number of vectors in any basis of S is the same and is called the **dimension** of S.

For the row and column space of a matrix the following property holds.

Theorem: The row and column space of a matrix A have the same dimension.

The **rank** of a matrix A, denoted rank(A), is the dimension of its row and column spaces. The **nullity** of a matrix A, denoted nullity(A), is the dimension of its null space.

It is easy to see that $\text{rank}(A^T) = \text{rank}(A)$. The rank and the nullity of a matrix have the following relation.

The Rank Theorem: If A is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n.$$

The fundamental theorem for invertible matrices can be now extended using some of these new notions.

<u>**The Fundamental Theorem for Invertible Matrices**</u>: Let A be an $n \times n$ matrix. The following are equivalent.

(a) A is invertible

(b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

(c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) A is a product of elementary matrices.

(f) $\text{rank}(A) = n$.

(g) $\text{nullity}(A) = 0$.

(h) The column vectors of A are linearly independent.

(i) The column vectors of A span $\mathbb{R}^n$.

(j) The column vectors of A form a basis for $\mathbb{R}^n$.

(k) The row vectors of A are linearly independent.

(l) The row vectors of A span $\mathbb{R}^n$.

(m) The row vectors of A form a basis for $\mathbb{R}^n$.

An application of the Rank Theorem and the Fundamental theorem gives the following.

<u>**Theorem**</u>: Let A be an $m \times n$ matrix. Then:

- (a) $\text{rank}(A^T A) = \text{rank}(A)$.
- (b) The $n \times n$ matrix $A^T A$ is invertible if and only if $\text{rank}(A) = n$.

Exercises:

1. Determine whether the following vectors form a basis for $\mathbb{R}^3$.

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$$