Practice Final Exam

Please note that the practice exam is meant to illustrate the format and the difficulty level of the actual exam. The types of problems and the represented topics on the actual exam may differ from those on the practice exam.

No calculators, closed notes and books!
Please write neatly and provide ample explanations. Answers without proper justification will not be counted. Check your work carefully. Cross out anything you don’t want to be graded.

1. [10] Evaluate the integrals:
   
   \( (a) \int \frac{\sin \frac{1}{3x^2}}{x} \, dx \)  
   \( (b) \int x \sec^2 x \, dx \)  
   \( (c) \int_1^3 \sqrt{x} \tan^{-1} \sqrt{x} \, dx \)  
   \( (d) \int \frac{x^2}{x^2 - 3x + 2} \, dx \).

2. [10]
   
   \( (a) \) Determine whether the following improper integral converges or diverges. Evaluate it if it converges.
   \( \int_0^{+\infty} \frac{dx}{\sqrt{x}(x + 1)} \).

   \( (b) \) A particle moves in a straight line with velocity \( v(t) = (t - \sqrt{t}) \) \( m/sec \). Find the distance traveled in the first four seconds.

3. [10] Find the area of the region enclosed by the curves \( x^2 = y \) and \( x = y - 2 \).

4. [10] Find the volume of the solid that results from rotating the region enclosed by the curves \( y = x^2 \) and \( y = x^3 \) about the line \( x = 1 \).

5. [10] Determine whether the following series converge absolutely, converge conditionally or diverge.
   
   \( (a) \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln k}{k} \)  
   \( (b) \sum_{k=1}^{\infty} \frac{\sin k}{k^3} \)  
   \( (c) \sum_{k=1}^{\infty} \frac{k^2 \sin^2 \left( \frac{1}{k} \right)}{k} \)  
   \( (d) \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \).

6. [10]
   
   \( (a) \) Find the Maclaurin series of the function \( f(x) = \frac{x}{1 - x^2} \) from that of the function \( 1/(1 - x) \).

   \( (b) \) Find the radius of convergence and the interval of convergence for the power series
   \( \sum_{k=1}^{\infty} \frac{x^k}{3^k(1 + k^2)}. \)

7. [10]
   
   \( (a) \) Solve the differential equation
   \( y' = (1 + y^2)x^2. \)

   \( (b) \) Solve the initial value problem
   \( y' = \frac{e^{-x} - e^x}{3 + 4y}, \quad y(0) = 1. \)
8. A first order differential equation is called a Bernoulli equation, if it can be written in
the form
\[ \frac{dy}{dx} + p(x)y = q(x)y^n, \quad \text{for } n \neq 0, 1. \]
If \( y = y(x) \) is a solution to a Bernoulli equation, define \( u = u(x) = [y(x)]^{1-n} \).
(a) Find a first-order linear differential equation that is satisfied by \( u \).
(b) Use your answer to part (a) to solve the initial value problem
\[ x\frac{dy}{dx} - y = -2xy^2, \quad y(1) = \frac{1}{2}. \]

9. [10]
(a) Find the general solution of the differential equation
\[ y'' + 2y' - 8y = 0 \]
(b) Solve the initial value problem
\[ 9y'' - 12y' + 4y = 0, \quad y(0) = 2, y'(0) = -1. \]

10. [10]
(a) Find the general solution of the differential equation
\[ y'' + 9y = t^2e^{3t} + 6. \]
(b) Show that \( y_1(t) = t^2 \) and \( y_2(t) = t^{-1} \) solve the homogeneous equation
\[ t^2y'' - 2y = 0, \quad t > 0. \]
(c) Use the method of variation of parameters and part (a) to find the general solution of
the inhomogeneous differential equation
\[ t^2y'' - 2y = 3t^2 - 1, \quad t > 0. \]