Economical tight examples for the biased Erdős–Selfridge theorem

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Abstract

A positional game is essentially a generalization of Tic-Tac-Toe played on a hypergraph $(V, \mathcal{F})$. A pivotal result in the study of positional games is the Erdős–Selfridge theorem, which gives simple criteria for the existence of a Breaker's winning strategy on a hypergraph $\mathcal{F}$. It has been shown that the Erdős–Selfridge theorem can be tight and that numerous extremal systems exist for that theorem. We focus on a generalization of the Erdős–Selfridge theorem proven by Beck for biased $(p : q)$ games, which we call the $(p : q)$–Erdős–Selfridge theorem. We show that for pn-uniform hypergraphs there is a *unique* extremal system for the $(p : q)$–Erdős–Selfridge theorem $(q \geq 2)$ when Maker must win in exactly n turns (i.e., as quickly as possible).

Key words: positional games, generalized Tic-Tac-Toe, Maker–Breaker games

1 Introduction

A positional game is a generalization of Tic-Tac-Toe played on a hypergraph $(V, \mathcal{F})$ where the vertices can be considered the "board" on which the game is played, and the edges can be thought of as the "winning sets." (In this paper we will only consider finite hypergraphs.) A *positional game* on $(V, \mathcal{F})$ is a two player game where at every turn each player alternately occupies a vertex from V. A *biased positional game* or a $(p : q)$ *positional game* on $(V, \mathcal{F})$ is a two player game where at every turn the first player occupies p vertices and then the second player occupies q vertices from V. The game is over when all vertices of $\mathcal{F}$ have been occupied. In a *strong positional game*, the first player to occupy all vertices of some edge $A \in \mathcal{F}$ wins. If at the end of play no edge is completely occupied by either player, that play is declared a draw.

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Normal 3×3 Tic-Tac-Toe is a strong positional game where the vertices of the hypergraph are the nine positions and the edges are the eight winning–lines. In a *Maker–Breaker* positional game, the first player, *Maker*, wins if she[1] occupies all vertices of some edge $A \in \mathcal{F}$, otherwise the second player, *Breaker*, wins. Therefore, by definition there are no draw plays in Maker–Breaker games. We say that a player P has a *winning strategy* if no matter how the other player plays, player P wins by following that winning strategy.

A pivotal result in the study of positional games is the Erdős–Selfridge theorem [3], which gives simple criteria based on a probabilistic intuition for the existence of an explicit Breaker's winning strategy on a hypergraph $\mathcal{F}$. It states that if

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2}, \tag{1}$$

then Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{F}$. In the case where $\mathcal{F}$ is n-uniform, condition (1) simplifies to $|\mathcal{F}| < 2^{n-1}$. Despite the simplicity of the theorem, it is extremely powerful and can be used to determine asymptotically tight breaking points for many games. One of the most impressive results stemming from the Erdős–Selfridge theorem is given by Beck [2]. In his paper, he considers the Maker–Breaker graph Ramsey game where Maker and Breaker take turns occupying edges of the complete graph K_n and Maker's goal is to occupy all the edges of any k-clique (complete subgraph on k vertices). Using his ingenious game theoretic second moment method, Beck shows that if $k \leq 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - \frac{10}{3} + o(1)$, then Maker has an explicit winning strategy. While on the other hand, by using the Erdős–Selfridge theorem, he shows that if $k \geq 2\log_2 n - 2\log_2\log_2 n + 2\log_2 e - 1 + o(1)$, then Breaker has an explicit winning strategy. This is clearly an amazing result as it shows that for large enough values of n, there are only three values of k for which we do not know who wins the Maker–Breaker graph Ramsey game.

In addition to the remarkable results that the Erdős–Selfridge theorem yields, it has been shown that the bound in the theorem is tight and that there are numerous n-uniform hypergraphs with exactly 2^{n-1} edges on which Maker has a winning strategy. We will call such hypergraphs *extremal systems* for the Erdős–Selfridge theorem because they have exactly the minimum number of edges allowable for Maker to possibly possess a winning strategy, and indeed Maker does have a winning strategy for the Maker–Breaker game played on these hypergraphs. In this paper we shall focus on a generalization of the Erdős–Selfridge theorem proven by Beck [1] for $(p : q)$ games, which we call

[1] In this paper, we will refer to Player 1 and Maker with feminine pronouns like "she" and "her," and we will refer to Player 2 and Breaker with masculine pronouns.

the $(p : q)$–Erdős–Selfridge theorem (or sometimes the *biased Erdős–Selfridge theorem*). The $(p : q)$–Erdős–Selfridge theorem states that if

$$\sum_{A \in \mathcal{F}} (q+1)^{-\frac{|A|}{p}} < \frac{1}{q+1} \tag{2}$$

then Breaker has an explicit winning strategy for the $(p : q)$-Maker–Breaker game played on $\mathcal{F}$. In the case where $\mathcal{F}$ is pn-uniform, condition (2) simplifies to $|\mathcal{F}| < (q+1)^{n-1}$. Along with this theorem, Beck also gave an example of a pn-uniform hypergraph $\mathcal{F}$ with $|\mathcal{F}| = (q+1)^{n-1}$ on which Maker has a winning-strategy, i.e., an extremal system for the $(p : q)$–Erdős–Selfridge theorem, thus showing that the theorem is tight. In this paper we will prove that if $q \geq 2$ and if we add the stipulation that Maker must win in exactly n turns (i.e., if Maker has an n-turn winning-strategy), then the extremal system given by Beck is unique.

The results of this paper were arrived at independently of Lu [4], in which he previously studied the extremal systems for both the biased and unbiased Erdős–Selfridge theorem. In Lu's paper, he examined the case where Maker has a winning strategy that allows her to win in the minimum number of turns, i.e., Maker has an *economical* winning strategy. He showed that there are numerous extremal systems for the Erdős–Selfridge theorem on which Maker has an economical winning strategy and he also investigated the economical extremal systems for the $(p : q)$–Erdős–Selfridge theorem. However, based on the conclusions that Lu reached, it seems that he may have unwittingly assumed that Breaker should always follow the Erdős–Selfridge strategy. Thus, some of the hypergraphs that he believed to be extremal systems for the biased and unbiased Erdős–Selfridge theorems were, in fact, not extremal systems, since Breaker could win the Maker–Breaker game played on those hypergraphs by using a strategy other than the Erdős–Selfridge strategy.

With respect to the extremal systems for the $(p : q)$–Erdős–Selfridge theorem, Lu reached the same conclusions as this paper up to (but not including) Claim 2. His technique was essentially to analyze Beck's proof of the $(p : q)$–Erdős–Selfridge theorem and use the fact that all of the inequalities in the proof must hold with equality when the hypergraph is an extremal system. In this paper, however, we reach those conclusions by simply appealing to the fact that an extremal system has the minimum number of edges possible for Maker to have a chance of winning.

In this paper we will prove that, in fact, there is a unique economical extremal system for the $(p : q)$–Erdős–Selfridge theorem when $q \geq 2$; and we will point out that the same extremal system is also the unique extremal system when Maker can take as many turns as she wants to win. Thus, the extremal system given by Beck in his original paper is the only extremal system for the $(p : q)$–

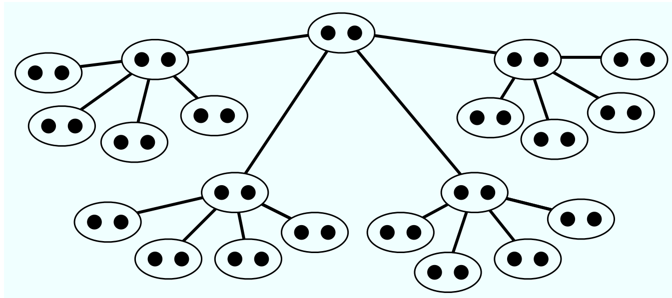

Fig. 1. An example of a 3-level, 4-ary tree where each node has two vertices, i.e., a $T(2,4,3)$. This is a 6-uniform hypergraph with 4^2 edges.

Erdős–Selfridge theorem when $q \geq 2$.

To explain the example given by Beck, we first consider the following generalization of a complete binary tree. A *complete n-level $(q+1)$-ary tree* is a generalization of a complete n-level binary tree, where each (non-leaf) node has $(q+1)$ children as a binary tree has two. Thus at level l of the $(q+1)$-ary tree there are $(q+1)^{l-1}$ nodes. The hypergraph we wish to consider can be derived from the complete n-level $(q+1)$-ary tree so that each node of the tree is identified with p distinct vertices of the hypergraph. Thus, the hypergraph has p times as many vertices as the tree has nodes. Whereas a tree-edge in the underlying tree connects two nodes of the tree, an edge in the hypergraph consists of all of the vertices from a path beginning at the root node and ending at a leaf node. Since there is a unique path from the root to any leaf, and since every path from the root to a leaf contains n nodes, and thus pn vertices, we can conclude that this is indeed a pn-uniform hypergraph with $(q+1)^{n-1}$ edges. Let us use $T(p,q+1,n)$ to denote the hypergraph just described. (Note that we could have defined a $T(p,q,n)$ to be the analogous hypergraph derived from a complete n-level *q-ary* tree; however for this paper we will invariably focus our attention on $(q+1)$-ary trees.) See Figure 1 for a drawing of a $T(2,4,3)$ hypergraph.

We can also define $T(p,q+1,n)$ inductively. Let $T(p,q+1,1)$ be a single (hyper)edge with p vertices in it. The hypergraph $T(p,q+1,n)$ is created by taking a set of p new vertices R, that will constitute a root node, along with $q+1$ vertex disjoint copies, T_i $(i=1,\ldots,q+1)$, of $T(p,q+1,n-1)$ so that each edge of $T(p,q+1,n)$ has the form $R \cup A$, where $A \in T_i$ for some $i \in \{1,\ldots,q+1\}$.

The winning-strategy that Maker has on $T(p,q+1,n)$ can be described as follows. First Maker occupies all p vertices from the root node. Then there are (essentially) $(q+1)$ disjoint $T(p,q+1,n-1)$'s left over. (Each $T(p,q+1,n-1)$ is rooted at level 2 in the original $T(p,q+1,n)$.) Breaker can choose his vertices from at most q of the $T(p,q+1,n-1)$'s. Thus there will always be a $T(p,q+1,n-1)$ in which Breaker has no vertices. Maker chooses her next p

vertices in the root of an unoccupied $T(p, q+1, n-1)$ and continues in that manner until she reaches a leaf node.

2 Main Theorem

Theorem 1 *Consider the $(p:q)$-Maker–Breaker game on a pn-uniform hypergraph $\mathcal{F}$, where $q \geq 2$. If there exists a Maker's winning strategy on $\mathcal{F}$ that takes exactly n turns, and if $|\mathcal{F}| = (q+1)^{n-1}$, then $\mathcal{F}$ is $T(p, q+1, n)$.*

Proof: We proceed by induction on n. The base case is $n = 1$. Since there is one edge of size p, Maker completely occupies it in one turn. Also note that $T(p, q+1, 1)$ is the hypergraph with one edge of size p. We assume the theorem is true for $k < n$ and we will show it is true for $k = n$. We will use the following lemma to assist us in our proof.

Lemma 2 *If $\mathcal{F}$ is a pn-uniform hypergraph that has the minimum number of edges for which Maker has an n-turn winning strategy in the $(p:q)$-Maker–Breaker game, then $|\cap_{A \in \mathcal{F}} A| = p$.*

(Note that the condition on the number of edges in the lemma can also be stated as "$\mathcal{F}$ is *minimal* with respect to edges.")

Proof of Lemma: If $|\cap_{A \in \mathcal{F}} A| > p$, then Breaker will pick one of the vertices in $\cap_{A \in \mathcal{F}} A$ for his first move and win the game. Therefore, we may assume $|\cap_{A \in \mathcal{F}} A| \leq p$. Suppose that $\mathcal{F}$ has the minimum number of edges for which Maker has an n-turn winning strategy. Let X_1 be the set of p vertices that Maker chooses during turn 1. Let $\mathcal{C} = \{A \in \mathcal{F} : A \supseteq X_1\}$ be the set of edges that contain X_1 and let $\mathcal{N} = \mathcal{F} \setminus \mathcal{C}$ be the set of edges that do not contain all of X_1. If $A \in \mathcal{N}$ then A cannot be completed in n turns because Maker occupied at most $p-1$ vertices from A in the first turn and can only occupy $p(n-1)$ more vertices from A in the remaining $n-1$ turns. If $\mathcal{N} \neq \emptyset$ then $|\mathcal{C}| < |\mathcal{F}|$. Since Maker is trying to win in n turns, she must win using an edge from $\mathcal{C}$, and Breaker will disregard the edges in $\mathcal{N}$. Thus Maker will win the game on $\mathcal{F}$ if and only if she wins the game restricted to $\mathcal{C}$. (See Figure 2.) Yet, if $|\mathcal{C}| < |\mathcal{F}|$ then Maker cannot win on $\mathcal{C}$ by the minimality of $|\mathcal{F}|$. Therefore, $\mathcal{N} = \emptyset$ and $|\cap_{A \in \mathcal{F}} A| = p$ completing the proof of our lemma. $\square$

Now we are assuming that the theorem is true for $k < n$, and we are showing that it holds for $k = n$. (See Figures 3 through 7 for pictures which highlight certain steps of the proof.) Let X_1 be the set of p vertices that Maker chooses in the first turn. By the proof of Lemma 2 we know that $|\cap_{A \in \mathcal{F}} A| = p$ and that it must be the case that $X_1 = \cap_{A \in \mathcal{F}} A$. Now let us consider $q+1$ hypothetical first moves by Breaker and use the responses by

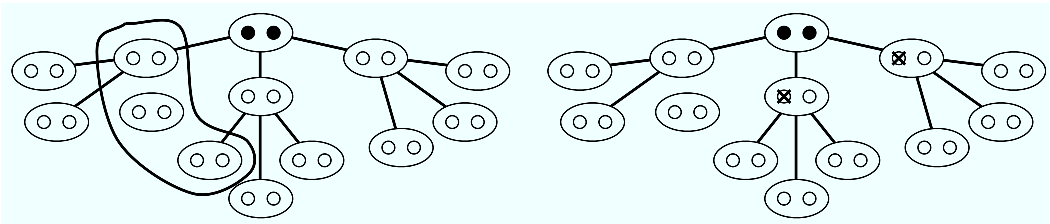

Fig. 2. The "circled" edge that does not contain Maker's first move is an element of $\mathcal{N}$. As far as Breaker is concerned, he only sees the hypergraph on the right, i.e., the edges in $\mathcal{C}$. (In this figure and in subsequent figures, Maker's moves are indicated by filled, black points and Breaker's moves are indicated by X's. Also, each figure depicts a $(2:2)$ game.)

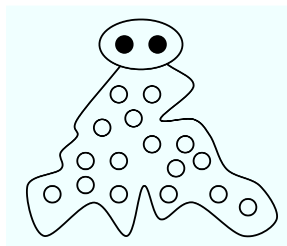

Fig. 3. Maker's first move is the root.

Maker's winning strategy to uncover the necessary $(q+1)$-ary tree-like structure of $\mathcal{F}$. Breaker begins by tentatively trying $Y_1^{(1)} = \{y_1^{(1)}, y_2^{(1)}, \ldots, y_q^{(1)}\}$ as his first move. Maker reveals to Breaker that, based on her winning strategy, she would respond by playing $X_2^{(1)} = \{x_1^{(1)}, x_2^{(1)}, \ldots, x_p^{(1)}\}$. From the proof of Lemma 2, Breaker knows that the only edges on which Maker can win are those that contain every vertex of every move that she makes. Thus, there will be a significant number of edges that contain every vertex of $X_2^{(1)}$, in particular, there will be a significant number of edges that contain $x_1^{(1)}$. Therefore, Breaker preempts Maker's response by including $x_1^{(1)}$ in place of $y_1^{(1)}$ in his first move and tries $Y_1^{(2)} = \{x_1^{(1)}, y_2^{(1)}, \ldots, y_q^{(1)}\}$. To this, Maker responds with $X_2^{(2)} = \{x_1^{(2)}, x_2^{(2)}, \ldots, x_p^{(2)}\}$. Similarly, there will be a significant number of edges A such that $A \supseteq X_2^{(2)}$, yet $A \cap Y_1^{(2)} = \emptyset$. Thus, Breaker repeatedly preempts Maker's previous responses by offering first moves in such a way that his ith attempt, $Y_1^{(i)} = \{x_1^{(1)}, \ldots, x_1^{(i-1)}, y_i^{(1)}, \ldots, y_q^{(1)}\}$, contains a vertex from each of Maker's previous $i-1$ responses $X_2^{(1)}, \ldots, X_2^{(i-1)}$. But Breaker can preempt at most q of Maker's responses, so he ends with the $(q+1)$th first move $Y_1^{(q+1)} = \{x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(q)}\}$ consisting entirely of vertices chosen by Maker, to which Maker responds with $X_2^{(q+1)}$.

We now consider what would happen if situation i was to occur. Let $\mathcal{F}(i) = \{A \setminus X_1 : A \in \mathcal{F}, A \cap Y_1^{(i)} = \emptyset\}$ be the set of partial edges that survive Breaker's first move in situation i. Let $\mathcal{C}(i) = \{A \in \mathcal{F}(i) : A \supseteq X_2^{(i)}\}$ be those surviving partial edges that contain all the vertices in Maker's second

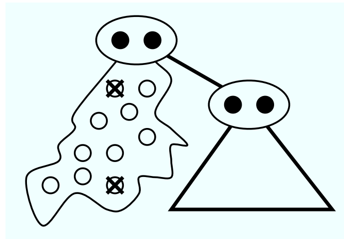

Fig. 4. By induction, no matter what Breaker's first move is, there will always be p other vertices that Maker can occupy so that $(q+1)^{n-2}$ alive edges contain all of those p vertices.

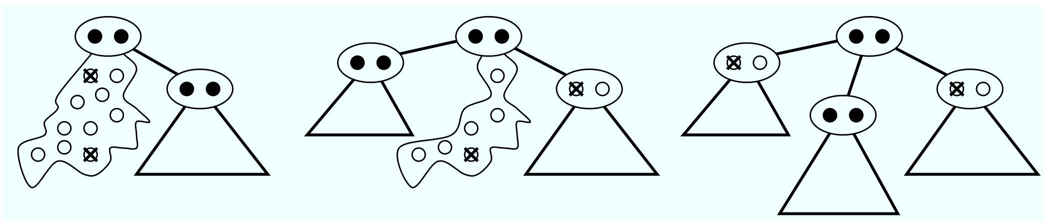

Fig. 5. By considering hypothetical Breaker and Maker moves, we can determine information about the structure of $\mathcal{F}$. For every Breaker move there is always a way for Maker to win.

move in situation i. Let $\mathcal{N}(i) = \mathcal{F}(i) \setminus \mathcal{C}(i)$ be those surviving partial edges that do not contain all of $X_2^{(i)}$. Since Maker must win in exactly n turns, we can show, by using an argument similar to the proof of Lemma 2, that she cannot win on any edges from $\mathcal{N}(i)$. Thus Breaker only has to block edges from $\mathcal{C}(i)$. Therefore, for all intents and purposes, both players are playing on $\mathcal{F}(i)$ restricted to $\mathcal{C}(i)$. However, Maker has a winning-strategy on $\mathcal{F}$, and both of her moves in turns one and two were made according to that strategy; therefore, after the first move is played by both players, Maker has a winning-strategy on $\mathcal{C}(i)$. Since Maker has a winning-strategy on $\mathcal{C}(i)$, and since $\mathcal{C}(i)$ is a $p(n-1)$-uniform hypergraph, the $(p:q)$–Erdős–Selfridge theorem implies that $|\mathcal{C}(i)| \geq (q+1)^{n-2}$. Of course these conclusions are true for $1 \leq i \leq q+1$.

Now we wish to show that if $i < j$, then $\mathcal{C}(i) \cap \mathcal{C}(j) = \emptyset$, or more formally, that $\mathcal{C}^*(i) \cap \mathcal{C}^*(j) = \emptyset$ where $\mathcal{C}^*(i) = \{A' \cup X_1 : A' \in \mathcal{C}(i)\}$ for $1 \leq i \leq q+1$, i.e., $\mathcal{C}^*(i)$ is the set of (full) edges in $\mathcal{F}$ that lead to the partial edges in $\mathcal{C}(i)$. Once we show that $\mathcal{C}^*(i) \cap \mathcal{C}^*(j) = \emptyset$ for $i < j$, then we will be able to conclude that there are $q+1$ edge disjoint $\mathcal{C}^*(i)$'s contained in $\mathcal{F}$ and therefore obtain an upper bound on the number of edges in each $\mathcal{C}^*(i)$.

Claim 1 *If $i < j$, then $\mathcal{C}^*(i) \cap \mathcal{C}^*(j) = \emptyset$.*

Proof of Claim 1: Let $A \in \mathcal{C}^*(i)$ be an arbitrary edge from $\mathcal{C}^*(i)$. By definition of $\mathcal{C}^*(i)$, we know that $A \supseteq X_2^{(i)}$ and $x_1^{(i)} \in X_2^{(i)}$. Thus every edge in $\mathcal{C}^*(i)$ contains the vertex $x_1^{(i)}$. Let $B \in \mathcal{C}^*(j)$ be an arbitrary edge from $\mathcal{C}^*(j)$. By definition of $\mathcal{C}^*(j)$, we know that $B \cap Y_1^{(j)} = \emptyset$. Since $j > i$, we have that

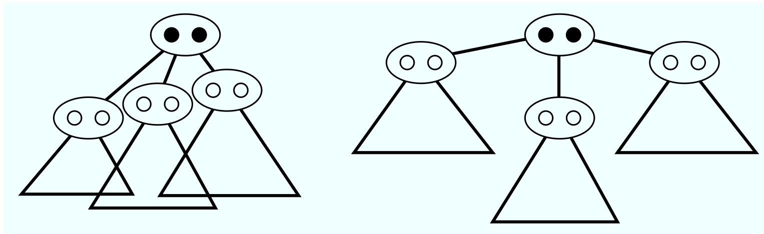

Fig. 6. We have determined that $\mathcal{F}$ is a root node connected to $(q+1)$ edge disjoint $T(p, q+1, n-1)$'s, but we need to show that the $T(p, q+1, n-1)$'s are vertex disjoint.

$x_1^{(i)} \in Y_1^{(j)}$, and therefore $x_1^{(i)} \notin B$. Thus every edge in $\mathcal{C}^*(j)$ does not contain the vertex $x_1^{(i)}$. Therefore it is impossible for an edge in $\mathcal{F}$ to be in $\mathcal{C}^*(i) \cap \mathcal{C}^*(j)$. $\square$

Since we now know that all of the $\mathcal{C}^*(i)$'s are edge disjoint we can conclude that $\mathcal{F}$ has at least as many edges as the sum of $|\mathcal{C}^*(i)|$'s. Together with the fact that $|\mathcal{C}^*(i)| \geq (q+1)^{n-2}$ for each $\mathcal{C}^*(i)$ we obtain the following inequalities

$$(q+1)^{n-1} = |\mathcal{F}| \geq \sum_{i=1}^{q+1} |\mathcal{C}^*(i)| \geq (q+1)(q+1)^{n-2}.$$

Yet this implies that $|\mathcal{C}^*(i)| = (q+1)^{n-2}$ for each $\mathcal{C}^*(i)$ since no $\mathcal{C}^*(i)$ can contain fewer than $(q+1)^{n-2}$ edges.

Now we recall that $|\mathcal{C}(i)| = |\mathcal{C}^*(i)| = (q+1)^{n-2}$, and that Maker has an $(n-1)$-turn winning-strategy on $\mathcal{C}(i)$ for each i, therefore by induction, each $\mathcal{C}(i)$ is a $T(p, q+1, n-1)$. Thus, so far we have concluded that $\mathcal{F}$ has a root node and $q+1$ edge disjoint $T(p, q+1, n-1)$'s connected to the root node, but we are not sure whether or not the $\mathcal{C}(i)$'s are vertex disjoint. (See Figure 6.) If we can show that the $\mathcal{C}(i)$'s are vertex disjoint, then we will have established that $\mathcal{F}$ is indeed $T(p, q+1, n)$, and we will have proved the theorem.

As we mentioned earlier, in his paper Lu reached the same conclusions that we have established so far. However, whereas we have simply appealed to the fact that an extremal system has the minimum number of edges possible for Maker to have a chance of winning, Lu essentially analyzed Beck's proof of the $(p : q)$–Erdős–Selfridge theorem and used the fact that all of the inequalities in the proof must hold with equality when the hypergraph is an extremal system. Claim 2 and the conclusion that $T(p, q+1, n)$ is the unique economical n-uniform extremal system for the $(p : q)$–Erdős–Selfridge theorem were not contained in Lu's paper.

Claim 2 *If $i \neq j$, then $V(\mathcal{C}(i)) \cap V(\mathcal{C}(j)) = \emptyset$.*

Proof of Claim 2: Assume, towards a contradiction, that $i \neq j$ and $V(\mathcal{C}(i)) \cap$

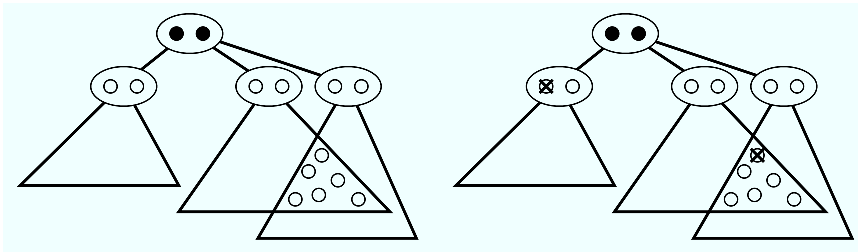

Fig. 7. Breaker occupies a vertex from the root of every other $\mathcal{C}(k)$ and the "highest" vertex from $V(\mathcal{C}(i)) \cap V(\mathcal{C}(j))$.

$V(\mathcal{C}(j)) \neq \emptyset$. Each $\mathcal{C}(k)$ is a $T(p, q+1, n-1)$, thus each $\mathcal{C}(k)$ has a root. Let $R(k) = \{c_1^{(k)}, \ldots, c_p^{(k)}\}$ be the root of $\mathcal{C}(k)$. Let $I = V(\mathcal{C}(i)) \cap V(\mathcal{C}(j))$ be the set of vertices in both $\mathcal{C}(i)$ and $\mathcal{C}(j)$. For $v \in I$, let $l_i(v)$ and $l_j(v)$ be the level of v in $\mathcal{C}(i)$ and the level of v in $\mathcal{C}(j)$ respectively. (Recall that $T(p, q+1, n-1)$ has a tree structure, thus the vertices in the root are at level 1, the vertices in the nodes that are children of the root are at level 2, $\ldots$, the vertices in nodes that are leaves are at level $n-1$.) Let $l(v) = \min\{l_i(v), l_j(v)\}$ be the minimum of the two levels associated with v. Now let $a \in I$ be such that $l(a) = \min_{v \in I} l(v)$, i.e., let a be a vertex with the smallest level amongst the vertices in the intersection of $\mathcal{C}(i)$ and $\mathcal{C}(j)$.

We know that Maker's first move is the root of $\mathcal{F}$, $X_1 = \cap_{A \in \mathcal{F}} A$. Let Breaker's first move be $Y_1 = \{a\} \cup \{c_1^{(k)} : k \neq i, k \neq j\}$. Since Breaker's first move contains a vertex from the root of every $\mathcal{C}(k)$ except $\mathcal{C}(i)$ and $\mathcal{C}(j)$, only edges from $\mathcal{C}(i)$ and $\mathcal{C}(j)$ are alive after the first turn. Maker's second turn is then forced to be either $X_2 = R(i)$ or $X_2 = R(j)$, because if Maker occupies neither $R(i)$ nor $R(j)$ for her second turn, then Breaker will take a vertex from $R(i)$ and vertex from $R(j)$ for his second turn and kill all remaining edges. (Notice that this requires $q \geq 2$.) Therefore, without loss of generality, let $X_2 = R(i)$.

We now claim that $R(i) \cap V(\mathcal{C}(j)) = \emptyset$ or $a \in R(j)$. If it is the case that $R(i) \cap V(\mathcal{C}(j)) \neq \emptyset$ and $a \notin R(j)$, then there is a vertex $b \in R(i) \cap V(\mathcal{C}(j)) \subseteq I$ such that $l(b) < l(a)$, which is a contradiction to our choice of a as a vertex in I with minimum possible level. (To see this, note that $l(b) = 1$ since $b \in R(i)$. Since $X_2 = R(i)$, then $R(i)$ was available to Maker, thus $a \notin R(i)$ and we are assuming that $a \notin R(j)$, thus $l(a) \geq 2$.) Therefore, we have that either $R(i) \cap V(\mathcal{C}(j)) = \emptyset$ or $a \in R(j)$.

If it is the case that $R(i) \cap V(\mathcal{C}(j)) = \emptyset$, then by using an argument like the one used in Lemma 2, we can conclude that if Maker is to win in n turns, then she must win in $\mathcal{C}(i)$. However, since Breaker already chose $a \in V(\mathcal{C}(i))$, the number of edges left in $\mathcal{C}(i)$ is less than $(q+1)^{n-2}$ and by the $(p:q)$–Erdős–Selfridge theorem, Maker cannot win in $\mathcal{C}(i)$ and therefore, cannot win in n turns.

If it is the case that $a \in R(j)$, then all edges of $\mathcal{C}(j)$ are killed by a since a is in the root. Once again $a \in V(\mathcal{C}(i))$ also, so the number of edges left in $\mathcal{C}(i)$ is less than $(q+1)^{n-2}$ and by the $(p:q)$–Erdős–Selfridge theorem, Maker cannot win in $\mathcal{C}(i)$ and therefore cannot win.

Therefore, by contradiction, it must be the case that $V(\mathcal{C}(i)) \cap V(\mathcal{C}(j)) = \emptyset$ if $i \neq j$. □

Thus we have established that $\mathcal{F}$ is a p-vertex root connected to $q+1$ edge and vertex disjoint $T(p, q+1, n-1)$'s, in other words, $\mathcal{F}$ is indeed $T(p, q+1, n)$. ■

We now mention that although this paper only proves that there is a unique *economical* extremal system for the $(p : q)$–Erdős–Selfridge theorem when $q \geq 2$, it is also true that the $T(p, q+1, n)$ is the unique extremal system when Maker is allowed to take more turns to win. The proof of the non-economical case is much more complicated and is contained in the second chapter of the dissertation [5] of Sundberg.

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