Economical Extremal Hypergraphs for the Erdős–Selfridge Theorem

Emily Heath
Department of Mathematics
University of Illinois
Urbana, IL 61801
eheath3@illinois.edu

Eric Sundberg
Mathematics Department
Occidental College
Los Angeles, CA 90041
sundberg@oxy.edu

April 4, 2016

Abstract

A positional game is essentially a generalization of tic-tac-toe played on a hypergraph $(V, \mathcal{H})$. A pivotal result in the study of positional games is the Erdős–Selfridge theorem, which gives a simple criterion for the existence of a Breaker's winning strategy on a finite hypergraph $\mathcal{H}$. It has been shown that the bound in the Erdős–Selfridge theorem can be tight and that numerous extremal hypergraphs exist that demonstrate the tightness of the bound. We call an extremal hypergraph economical if it is n-uniform and Maker has an n-turn winning strategy on that hypergraph. While characterizing all extremal hypergraphs for the Erdős–Selfridge theorem is still an open problem, we make progress on this problem by giving two distinct characterizations of the economical extremal hypergraphs for the Erdős–Selfridge theorem: one of a theoretical nature, and one of a more practical nature.

1 Introduction

A positional game is a generalization of tic-tac-toe played on a hypergraph $(V, \mathcal{H})$ where the vertices can be considered the "board" on which the game is played, and the edges can be thought of as the "winning sets." (In this paper we will only consider finite hypergraphs.) A *positional game* on $(V, \mathcal{H})$ is a two-player game where at every turn each player alternately occupies a previously unoccupied vertex from V. In a *strong positional game*, the first player to occupy all vertices of some edge $A \in \mathcal{H}$ wins. If at the end of play no edge is completely occupied by either player, that play is declared a draw. Normal 3×3 tic-tac-toe is a strong positional game where the vertices of the hypergraph are the nine positions and the edges are the eight winning lines. In a *Maker–Breaker* positional game, the first player, *Maker*, wins if she[1] occupies all vertices of some edge $A \in \mathcal{H}$, otherwise the second player, *Breaker*,

[1] In this paper, we will refer to Maker with feminine pronouns such as "she" and "her," and we will refer to Breaker with masculine pronouns.

wins. Therefore, by definition, there are no draw plays in Maker–Breaker games. We say that a player P has a *winning strategy* if no matter how the other player plays, player P wins by following that winning strategy. It is well-known that in a Maker–Breaker game, exactly one player has a winning strategy. Please note that in this paper, we will use $\mathcal{H}$ both to denote the whole hypergraph $(V, \mathcal{H})$ and to denote just its set of edges, where the appropriate interpretation should be understood from the context.

A pivotal result in the study of positional games is the Erdős–Selfridge theorem [4]. In their paper, Erdős and Selfridge introduced the idea of transforming a probabilistic argument into a completely deterministic and efficient potential-based strategy for positional games. Their theorem gives a simple criterion for the existence of an explicit Breaker's winning strategy on a hypergraph $\mathcal{H}$. It states that if

$$\sum_{A \in \mathcal{H}} 2^{-|A|} < \frac{1}{2}, \tag{1}$$

then Breaker has an explicit winning strategy for the Maker–Breaker game played on $\mathcal{H}$. In the case where $\mathcal{H}$ is n-uniform, condition (1) simplifies to $|\mathcal{H}| < 2^{n-1}$. Despite its simplicity, the Erdős–Selfridge theorem can be used to determine the correct order of magnitude for the breaking points (i.e., the value of n at which a game switches from being a win for Maker to a win for Breaker) of many games. Moreover, it laid the groundwork for using potential-based strategies in positional game theory. These potential-based strategies play a key role in determining the asymptotically exact breaking points for many games where such a result is known. See Beck [2].

In addition to the remarkable results that stem from their theorem, Erdős and Selfridge provided an example of an n-uniform hypergraph with exactly 2^{n-1} edges on which Maker has a winning strategy, thus, proving that the bound in their theorem is tight. Let us call a hypergraph $\mathcal{H}$ an *extremal* hypergraph for the Erdős–Selfridge theorem if $\sum_{A \in \mathcal{H}} 2^{-|A|} = \frac{1}{2}$ and Maker has a winning strategy on $\mathcal{H}$.

In [1], Beck proved a generalization of the Erdős–Selfridge theorem for biased games, i.e., games where Maker occupies p vertices per turn and Breaker occupies q vertices per turn, which we call the $(p : q)$-Erdős–Selfridge theorem. Beck's theorem states that if

$$\sum_{A \in \mathcal{H}} (q+1)^{-\frac{|A|}{p}} < \frac{1}{q+1}, \tag{2}$$

then Breaker has an explicit winning strategy for the biased game played on $\mathcal{H}$ where Maker and Breaker occupy p and q vertices per turn, respectively. In the case where $\mathcal{H}$ is pn-uniform, condition (2) simplifies to $|\mathcal{H}| < (q+1)^{n-1}$. Along with this theorem, Beck also gave an example of a pn-uniform hypergraph $\mathcal{H}$ with $|\mathcal{H}| = (q+1)^{n-1}$ on which Maker has a winning strategy, i.e., an extremal hypergraph for the $(p : q)$-Erdős–Selfridge theorem, thus, showing that the bound in the theorem is tight. It should be noted that when $p = q = 1$, the $(p : q)$-Erdős–Selfridge theorem specializes to the Erdős–Selfridge theorem, and the extremal hypergraph Beck described also specializes to an extremal hypergraph for the Erdős–Selfridge theorem which is different from the extremal hypergraph described by Erdős and Selfridge in [4]. The extremal hypergraph given by Beck in [1] can be described by taking a complete

n-level, $(q+1)$-ary tree[2] and associating p distinct vertices with each node in the tree. Each (hyper)edge is composed of the pn vertices associated with the n nodes on a root-to-leaf path in the tree. Thus, the hypergraph is pn-uniform and contains exactly $(q+1)^{n-1}$ (hyper)edges. Maker's winning strategy can be described as follows: during turn i, Maker occupies the p vertices associated with a node N_i in level i, so that the subtree rooted at N_i does not contain any of Breaker's vertices at that moment. We note that Maker's strategy produces a win for Maker in n turns, which is the minimum number of turns in which Maker can possibly win. We call such a winning strategy an *economical winning strategy*, since every vertex occupied by Maker up to turn n is contained in the edge she occupies during turn n. We call an extremal hypergraph on which Maker has an economical winning strategy an *economical extremal hypergraph* (EEH). Thus, the hypergraph described by Beck is an EEH.

In [5], Lu studied economical extremal hypergraphs for both the Erdős–Selfridge theorem and the $(p:q)$-Erdős–Selfridge theorem. Lu described a way to generalize the EEH for the Erdős–Selfridge theorem given by Beck in order to produce a rich family of n-uniform hypergraphs. We describe Lu's construction below, but first we describe a more relevant modification of Lu's construction. Let T be a complete binary tree with n-levels, i.e., a rooted binary tree with exactly 2^{n-1} leaves all at a distance of $n-1$ from the root. There is a natural partial order on the nodes of T such that $N_i \leq N_j$ if N_i is a node on the path between N_j and the root. If $N_i < N_j$, then we say that N_i is an *ancestor* of N_j and N_j is a *descendant* of N_i. If $N^p < N$ and N^p is adjacent to N, then N^p is the *parent* of N. If N and N' share the same parent, then N and N' are *siblings*.

We say that a labeling $L : V(T) \to \mathbb{N}$ is a *good labeling* if it is a labeling of T such that:

1. if N and N' are siblings, then $L(N) \neq L(N')$;
2. if $N_i < N_j$, then $L(N_i) \neq L(N_j)$;
3. if N_i and $N_i{}'$ are siblings and N_j and $N_j{}'$ are siblings, then $L(N_i) = L(N_j)$ (if and) only if $L(N_i{}') = L(N_j{}')$.

We say that a hypergraph $\mathcal{H}$ is *realized* by a labeling L of T if each edge in $\mathcal{H}$ is precisely the labels on the nodes of a root-to-leaf path in T, i.e., $\{L(N_1), L(N_2), \ldots, L(N_n)\}$ is an edge in $\mathcal{H}$ if and only if $N_1, N_2, \ldots, N_n$ are the nodes of root-to-leaf path in T. Lu studied hypergraphs that are realized by labelings which satisfy the following two properties:

(L1) if N and N' are siblings, then $L(N) \neq L(N')$;

(L2) if $N_i < N_j$ and N_i' is the sibling of N_i, then $L(N_i) \neq L(N_j)$ and $L(N_i') \neq L(N_j)$.

We will call such labelings *Lu labelings*. Lu mistakenly believed that any hypergraph realized by a Lu labeling would necessarily be an EEH, however, Properties (L1) and (L2) do not even guarantee that $\mathcal{H}$ is an extremal hypergraph for the Erdős–Selfridge theorem. (See the hypergraph on the left in Figure 1.) In Chapter 6 of [3], Beck amended Lu's construction by replacing Property (L2) with Properties 2 and 3 of a good labeling. In this paper, we will say that a hypergraph $\mathcal{H}$ is a *tumbleweed* if it is realized by a good labeling L of T. Beck

[2] Our interpretation of complete n-level, $(q+1)$-ary tree is that the root is in level 1, and all $(q+1)^{n-1}$ leaves are in level n.

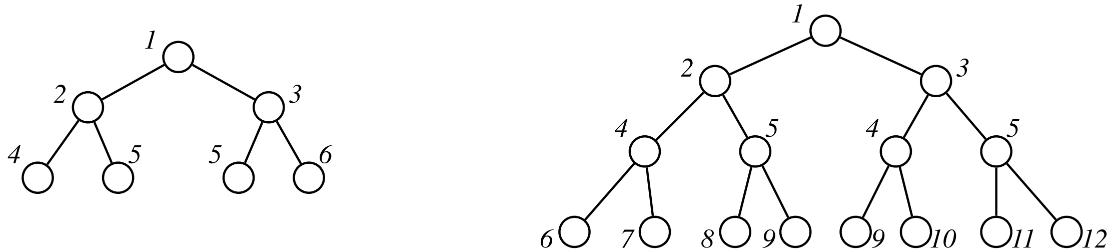

Figure 1: Two hypergraphs realized by Lu labelings on complete binary trees. These labelings satisfy Properties 1 and 2 (but not 3) of a good labeling. The hypergraph on the left is *not* an extremal hypergraph for the Erdős–Selfridge theorem. (Assuming Maker occupies label 1 for her first move, Breaker can win by occupying label 5 for his first move.) The hypergraph on the right is an extremal hypergraph for the Erdős–Selfridge theorem, but is not an EEH. (Assuming Maker occupies label 1 for her first move, Breaker can prevent Maker from winning within 4 moves by occupying label 9 for his first move.)

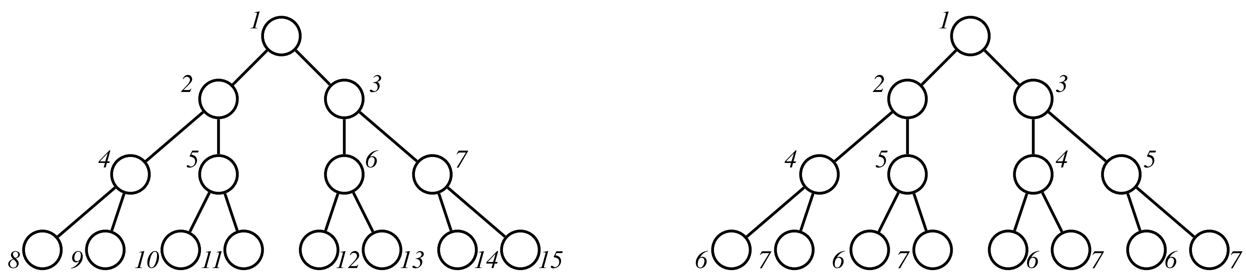

Figure 2: Two good labelings which realize EEHs. On the left is the binary tree extremal hypergraph given by Beck in [1] and on the right is the original extremal hypergraph given by Erdős and Selfridge in [4].

proved that all tumbleweeds are extremal hypergraphs for the Erdős–Selfridge theorem, since Property 3 provides a simple pairing strategy win for Maker. Specifically, Maker occupies the label of the root node for her first turn. Then for subsequent turns, if Breaker occupies $L(N)$ during turn $i-1$, Maker responds with $L(N')$, where N' is the sibling of N, during turn i, for $i \geq 2$. We call Maker's pairing strategy, the *sibling strategy.* The original extremal hypergraph provided by Erdős and Selfridge (which is also an EEH) and the EEH given by Beck in [1] are both examples of tumbleweeds. (See Figure 2.)

While every tumbleweed is certainly an extremal hypergraph for the Erdős–Selfridge theorem, Maker's winning sibling strategy is almost never an economical winning strategy, because it essentially allows Breaker to determine the length of the game. For example, if Maker uses the sibling strategy, Breaker can force Maker's win to be delayed until the last label is occupied. However, there may be other winning strategies available to Maker depending on the particular tumbleweed on which the game is played. For example, Maker has an economical winning strategy on the EEH originally given by Beck. Thus, we ask, "for each tumbleweed $\mathcal{H}$, does there exist an economical winning strategy for Maker on $\mathcal{H}$?"

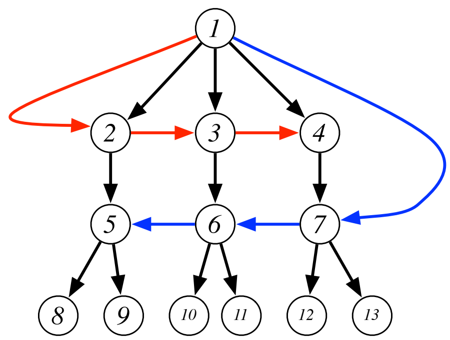

Figure 3: Two 4-uniform non-economical extremal hypergraphs for the Erdős–Selfridge theorem which are not derived from a labeling of a binary tree. The vertices are the labels on the nodes. The edges are the paths indicated by the arrows, i.e., the six downward black paths, the red (mostly) horizontal path $\{1, 2, 3, 4\}$, and the blue (mostly) horizontal path $\{1, 5, 6, 7\}$.

In Chapter 6 of [3], Beck answers this question in the negative by providing an example of an n-uniform ($n \geq 5$) tumbleweed where Breaker can force Maker to occupy at least $(2^{n-4} + n - 1)$ vertices of $\mathcal{H}$ in order for Maker to win. The example was originally described by A.J. Sanders in a manuscript from 2004.

While Beck revealed that every tumbleweed is an extremal hypergraph for the Erdős–Selfridge theorem and not every tumbleweed is an EEH, in this paper, we show that every EEH is a tumbleweed. In fact, we go further, since *we characterize all economical extremal hypergraphs for the Erdős–Selfridge theorem.*

The problem of characterizing all extremal hypergraphs for the Erdős–Selfridge theorem is still a wide open problem. There are examples of labelings which do not satisfy Property 3 of a good labeling, yet still realize an extremal hypergraph for the Erdős–Selfridge theorem. (See the hypergraph on the right in Figure 1.) There are also examples of extremal hypergraphs for the Erdős–Selfridge theorem which do not arise from a labeling of a binary tree. (See Figure 3 for two similar examples given by Sanders and Lu.) However, Sundberg proved in [6] that there is a unique EEH for the $(p : q)$-Erdős–Selfridge theorem when $q \geq 2$, namely, the original $(q+1)$-ary tree example given by Beck in [1]. Moreover, Sundberg later proved in [7] that even when we do not force Maker to win as quickly as possible, the same $(q+1)$-ary tree example is also the unique extremal hypergraph for the $(p : q)$-Erdős–Selfridge theorem when $q \geq 2$. Thus, the problems of characterizing the extremal hypergraphs for the $(p : q)$-Erdős–Selfridge theorem ($q \geq 2$) and characterizing the *economical* extremal hypergraphs for the Erdős–Selfridge are now settled.

The remainder of the paper is organized as follows. In Section 2, we state some fundamental definitions and lemmas. In Section 3, we prove some fundamental theorems about tumbleweeds that allow us to characterize all economical extremal hypergraphs for the Erdős–Selfridge theorem, in particular, Theorem 14. In Section 4, we prove two distinct characterizations of the economical extremal hypergraphs for the Erdős–Selfridge theorem, namely, Theorem 18 and Corollary 25.

2 Preliminaries

In this section, we state and prove some useful lemmas.

We begin by encouraging the reader to review the definitions of *good labeling*, *realized*, and *tumbleweed* from Section 1. The properties of a good labeling give us the following lemmas.

Lemma 1 *If $\mathcal{H}$ is a tumbleweed realized by a good labeling L of a complete n-level binary tree T, then $\mathcal{H}$ is n-uniform.*

Lemma 2 *If $\mathcal{H}$ is an n-uniform tumbleweed, then $\mathcal{H}$ has exactly 2^{n-1} distinct edges.*

Lemma 3 *Let L be a good labeling on a complete binary tree T. Let N be a node in T, and let T_N be the subtree rooted at N. Then L restricted to T_N is also a good labeling. Thus, the hypergraph $\mathcal{H}_N$ realized by L restricted to T_N is a tumbleweed.*

Proof of Lemma 3 If N_i and N_i' are siblings in T_N, then they are siblings in T. Thus, Properties 1 and 3 still hold. If $N_i < N_j$ in T_N, then $N_i < N_j$ in T. Thus, Property 2 also holds. $\square$

Property 3 of a good labeling allows us to define sibling vertices for a tumbleweed relative to a given good labeling L. Namely, let $\mathcal{H}$ be a tumbleweed, and let L be a good labeling which realizes $\mathcal{H}$. We say that $x, x' \in V(\mathcal{H})$ are *siblings in $\mathcal{H}$ relative to L* if there exist nodes $N, N' \in V(T)$ such that N and N' are siblings in T and $L(N) = x$ and $L(N') = x'$. (Figure 4 shows two distinct labelings which realize the same tumbleweed. Notice that labels 4 and 5 are siblings relative to both labelings. In Section 3, we prove Corollary 15 which states that if x and x' are siblings relative to some good labeling which realizes $\mathcal{H}$, then x and x' are siblings relative to any good labeling which realizes $\mathcal{H}$.) We now state a useful lemma that follows from Properties 2 and 3 of a good labeling.

Lemma 4 *Let $\mathcal{H}$ be an n-uniform tumbleweed, and let L be a good labeling on a complete n-level binary tree T which realizes $\mathcal{H}$. If x and x' are siblings in $\mathcal{H}$ relative to L, then no edge in $\mathcal{H}$ contains both x and x'.*

Proof of Lemma 4: Let N and N' be a pair of siblings in $V(T)$ such that $L(N) = x$ and $L(N') = x'$. Assume towards a contradiction that there is an edge $A \in \mathcal{H}$ which contains x and x'. Then this edge must correspond to a root-to-leaf path in T, say $N_1, N_2, \ldots, N_n$, in which $L(N_i) = x$ and $L(N_j) = x'$ for some $\{i, j\} \subseteq \{1, \ldots, n\}$ where, without loss of generality, $i < j$. Let N_j' be the sibling of N_j in T. Since L is a good labeling, Property 3 implies $L({N_j}') = x$. But $N_i < {N_j}'$ in T, and $L(N_i) = L({N_j}') = x$, which is a contradiction to Property 2. Therefore, if x and x' are siblings in $\mathcal{H}$ relative to L, then no edge in $\mathcal{H}$ contains both x and x'. $\square$

For a hypergraph $\mathcal{H}$ and vertex $x \in V(\mathcal{H})$, let $\mathcal{H}(x) = \{A \in \mathcal{H} : x \in A\}$ be the set of edges which contain x, and let $\mathcal{H}(\overline{x}) = \mathcal{H} - \mathcal{H}(x)$ be the set of edges which do not contain x.

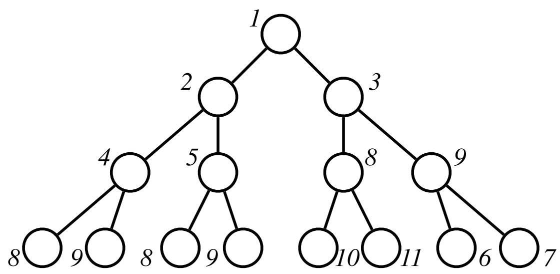

Figure 4: Two distinct good labelings which realize the same tumbleweed $\mathcal{H}$.

Lemma 5 *If x and x' are siblings in a tumbleweed $\mathcal{H}$ relative to a good labeling L, then $\mathcal{H}(x) \cap \mathcal{H}(x') = \emptyset$ and $|\mathcal{H}(x)| = |\mathcal{H}(x')|$.*

Proof of Lemma 5: Let $\mathcal{H}$ be a tumbleweed realized by a good labeling L of T. Let x and x' be siblings in $\mathcal{H}$ relative to L. By Lemma 4, $\mathcal{H}(x)$ and $\mathcal{H}(x')$ are disjoint. Every labeled root-to-leaf path which contains x must pass through a node of T which is labeled x. Thus, the number of edges containing x can be calculated by summing the number of leaves in each subtree T_N of T rooted at a node N which is labeled x. That is,

$$|\mathcal{H}(x)| = \sum_{N:L(N)=x} (\#\text{ of leaves in } T_N)$$

But since x and x' are siblings in $\mathcal{H}$ relative to L, if $L(N) = x$, then $L(N') = x'$ where N' is the sibling of N in T. Since N and N' are siblings and T is a complete binary tree, then

$$(\#\text{ of leaves in } T_N) = (\#\text{ of leaves in } T_{N'}),$$

which gives us

$$|\mathcal{H}(x)| = \sum_{N:L(N)=x} (\#\text{ of leaves in } T_N) = \sum_{N':L(N')=x'} (\#\text{ of leaves in } T_{N'}) = |\mathcal{H}(x')|.$$

□

Lemma 6 *Let $\mathcal{H}$ be a tumbleweed, and let $x, y \in V(\mathcal{H})$. If $x \neq y$, then $\mathcal{H}(x) \neq \mathcal{H}(y)$.*

Proof of Lemma 6: Let $\mathcal{H}$ be a tumbleweed and let $x, y \in V(\mathcal{H})$ such that $x \neq y$. Let L be a good labeling on a complete binary tree T which realizes $\mathcal{H}$. Let $A \in \mathcal{H}(x) \cap \mathcal{H}(y)$. (If $\mathcal{H}(x) \cap \mathcal{H}(y) = \emptyset$, then we are already done.) Let P be the root-to-leaf path in T which realizes A when T is labeled with L. Without loss of generality, x appears as a label on a node closer to the root of T than y does, in which case we can obtain $A' \in \mathcal{H}(x) - \mathcal{H}(y)$ by following P until we reach the parent of the node labeled y. Then we select the sibling of the node labeled y to ensure that y' (the sibling of y relative to L) is in our edge A' and continue our path P' until we reach a leaf. (See Figure 5.) Once we know $y' \in A'$, then Lemma 4 guarantees that $y \notin A'$. Since $x \in A'$, then $A' \in \mathcal{H}(x) - \mathcal{H}(y)$, as desired. □

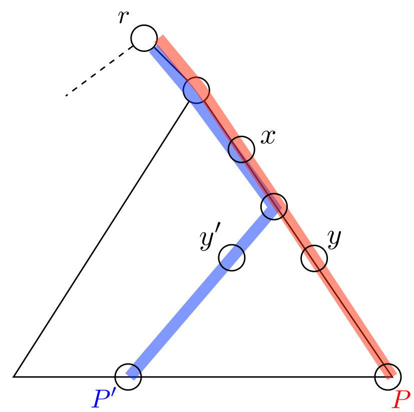

Figure 5: The path P on the right realizes the edge $A \in H(x) \cap H(y)$, while the path P' realizes the edge $A' \in H(x) - H(y)$.

If $\mathcal{H}$ is a hypergraph such that $|\cap_{A\in\mathcal{H}} A| = 1$, and $\{r\} = \cap_{A\in\mathcal{H}} A$, then we say r is a *root vertex* of $\mathcal{H}$. Throughout Sections 3 and 4 we will use the notation $\mathcal{H}_1 = \{A - r : A \in \mathcal{H}\}$, where r is the root of $\mathcal{H}$.

Lemma 7 *If $\mathcal{H}$ is a tumbleweed, then $\mathcal{H}$ has a root vertex r. If L is a good labeling on a complete binary tree T which realizes $\mathcal{H}$, then the root node of T is labeled r and is the only node labeled r.*

Proof of Lemma 7: Let $\mathcal{H}$ be a tumbleweed, and let L be a good labeling on a complete binary tree T which realizes $\mathcal{H}$. Let r be the label on the root node of T. Since every labeled root-to-leaf path contains r, we know that $|\cap_{A\in\mathcal{H}} A| \geq 1$. Let x be an arbitrary vertex in $\mathcal{H}$ such that $x \neq r$. Since $x \neq r$, then x has a sibling x' relative to L. Lemma 4 tells us that no edge contains both x and x'. Thus, if $A \in \mathcal{H}(x')$, then $x \notin A$, which implies that $x \notin \cap_{A\in\mathcal{H}} A$. Therefore $\cap_{A\in\mathcal{H}} A = \{r\}$, i.e., r is the root vertex of $\mathcal{H}$. Property 2 of a good labeling implies that no other node in T is labeled with r. $\square$

If $\mathcal{H}$ is an n-uniform tumbleweed realized by a good labeling L on a complete n-level binary tree T, we say that x and x' are a *heavy sibling pair* in $\mathcal{H}$ relative to L if x and x' are siblings in $\mathcal{H}$ relative to L, $\mathcal{H}(x)$ and $\mathcal{H}(x')$ partition $\mathcal{H}$, and $|\mathcal{H}(x)| = |\mathcal{H}(x')| = 2^{n-2}$. (In Figure 2, notice that every pair of siblings relative to the labeling on the right is a heavy sibling pair relative to that labeling.)

Lemma 8 *Let $\mathcal{H}$ be an n-uniform tumbleweed and let x be a vertex in $\mathcal{H}$ which is not the root vertex. If $|\mathcal{H}(x)| \geq 2^{n-2}$, then for every good labeling L which realizes $\mathcal{H}$, x and its sibling x' relative to L form a heavy sibling pair relative to L. In particular, $|\mathcal{H}(x)| = 2^{n-2}$.*

Proof of Lemma 8: Let $\mathcal{H}$ be an n-uniform tumbleweed and let $x \in V(\mathcal{H})$ satisfy $|\mathcal{H}(x)| \geq 2^{n-2}$ and x is not the root vertex. Let L be an arbitrary good labeling which realizes $\mathcal{H}$, and let x' be the sibling of x relative to L. By Lemma 5, $\mathcal{H}(x) \cap \mathcal{H}(x') = \emptyset$, thus, $|\mathcal{H}(x) \cup \mathcal{H}(x')| =$

$|\mathcal{H}(x)| + |\mathcal{H}(x')|$. Moreover, by Lemma 5, $|\mathcal{H}(x)| = |\mathcal{H}(x')|$, and since $|\mathcal{H}(x)| \geq 2^{n-2}$, then $|\mathcal{H}(x) \cup \mathcal{H}(x')| \geq 2(2^{n-2})$. But $\mathcal{H}(x) \cup \mathcal{H}(x') \subseteq \mathcal{H}$, thus, $|\mathcal{H}(x) \cup \mathcal{H}(x')| \leq |\mathcal{H}| = 2^{n-1}$. Therefore, $|\mathcal{H}(x)| = |\mathcal{H}(x')| = 2^{n-2}$, and we can conclude that x and x' are a heavy sibling pair relative to L. $\square$

We say that x and y are a *heavy pair* in an n-uniform tumbleweed $\mathcal{H}$ if $\mathcal{H}(x)$ and $\mathcal{H}(y)$ partition $\mathcal{H}$ and $|\mathcal{H}(x)| = |\mathcal{H}(y)| = 2^{n-2}$. Clearly, if x and x' are a heavy sibling pair of $\mathcal{H}$ relative to a good labeling L, then x and x' are a heavy pair in $\mathcal{H}$. We will show the converse, namely, heavy pairs must be siblings in every good labeling which realizes $\mathcal{H}$. Therefore, after we prove Lemma 9, we will use the terms heavy pair and heavy sibling pair interchangeably.

Lemma 9 *Let $\mathcal{H}$ be a n-uniform tumbleweed and let x and y be a heavy pair in $\mathcal{H}$. If L is a good labeling which realizes $\mathcal{H}$, then x and y are a heavy sibling pair relative to L.*

Proof of Lemma 9: Let $\mathcal{H}$ be a n-uniform tumbleweed and let x and y be a heavy pair in $\mathcal{H}$. Let L be an arbitrary good labeling on a complete n-level binary tree T which realizes $\mathcal{H}$. Let x' be the sibling of x relative to L. Since $|\mathcal{H}(x)| = 2^{n-2}$, Lemma 8 implies that x and x' form a heavy sibling pair relative to L. Since x and y are a heavy pair in $\mathcal{H}$, then $\mathcal{H}(x)$ and $\mathcal{H}(y)$ partition $\mathcal{H}$. Since x and x' are a heavy sibling pair, then $\mathcal{H}(x)$ and $\mathcal{H}(x')$ also partition $\mathcal{H}$. These two partitions of $\mathcal{H}$ allow us to conclude that $\mathcal{H}(y) = \mathcal{H}(x')$. Lemma 6 allows us to conclude that $y = x'$. $\square$

We now state some lemmas and definitions related to economical extremal hypergraphs for the Erdős–Selfridge theorem.

Lemma 10 (Root Vertex Lemma) *If $\mathcal{H}$ is an economical extremal hypergraph for the Erdős-Selfridge theorem, then $|\cap_{A\in\mathcal{H}} A| = 1$, i.e., $\mathcal{H}$ has a root vertex.*

Proof of Lemma 10: Let $\mathcal{H}$ be an n-uniform economical extremal hypergraph for the Erdős–Selfridge theorem. If $|\cap_{A\in\mathcal{H}} A| > 1$, then Breaker will pick one of the vertices in $\cap_{A\in\mathcal{H}} A$ (not occupied by Maker during her first turn) for his first move and win the game. Thus, $|\cap_{A\in\mathcal{H}} A| \leq 1$. Suppose $|\cap_{A\in\mathcal{H}} A| = 0$. Let x_1 be Maker's first move according to her economical winning strategy. We know that if $A \in \mathcal{H}(\overline{x_1})$, then Maker cannot complete A in n turns, because Maker can occupy at most $n-1$ of the vertices in A in her next $n-1$ turns. Therefore, since Maker has an economical winning strategy, Maker must win the game using an edge from $\mathcal{H}(x_1)$. Thus, Maker must be able to win the game restricted to $\mathcal{H}(x_1)$. However, since $|\cap_{A\in\mathcal{H}} A| = 0$, no vertex is contained in every edge of $\mathcal{H}$, thus, $|\mathcal{H}(x_1)| < |\mathcal{H}|$. But then Maker cannot win the game on $\mathcal{H}(x_1)$, as $|\mathcal{H}|$ is the minimum number of edges needed for Maker to have an economical winning strategy. Therefore, $|\cap_{A\in\mathcal{H}} A| = 1$. $\square$

Lemma 11 *In any Maker winning strategy on an economical extremal hypergraph $\mathcal{H}$ for the Erdős–Selfridge theorem, Maker must occupy the root vertex of $\mathcal{H}$ for her first move.*

We say that a tumbleweed $\mathcal{H}$ has the *Single Label Property (SLP)* if for each vertex $v \in V(\mathcal{H})$, there exists a good labeling of a complete binary tree T which realizes $\mathcal{H}$ in which v appears as a label exactly once. (The tumbleweed in Figure 4 has the SLP. The labels 1, 2, 3, 4, 5, 6, 7, 10, 11 each appear as a label exactly once in the labeling on the left, and the labels 8 and 9 each appear as a label exactly once in the labeling on the right.) In Section 4, we prove that a hypergraph $\mathcal{H}$ is an n-uniform economical extremal hypergraph for the Erdős–Selfridge theorem if and only if $\mathcal{H}$ is an n-uniform tumbleweed with the Single Label Property.

3 Structure of Tumbleweeds

In this section, we prove Theorem 14 which states that any labeling which realizes a tumbleweed is necessarily a good labeling. Theorem 14 is used to help prove our characterizations of the economical extremal hypergraphs for the Erdős–Selfridge theorem in Section 4.

Lemma 12 *Let $\mathcal{H}$ be an n-uniform tumbleweed, and let L be a good labeling on a complete n-level binary tree T which realizes $\mathcal{H}$. Let x and x' be a heavy sibling pair in $\mathcal{H}$ relative to L. Prune and contract T to give T_x by deleting all subtrees in T rooted at nodes labeled x' then contracting the tree edges in T between each node N labeled x and its parent node N^p and keep the name N^p for the resulting contracted node. (See Figure 6.) Let $L_x : V(T_x) \to \mathbb{N}$ be the labeling of $V(T_x)$ such that $L_x(N) = L(N)$ for all $N \in V(T_x)$. Then L_x is a good labeling on T_x which realizes an $(n-1)$-uniform tumbleweed which we call $\mathcal{H}_x$; moreover, $\mathcal{H}_x = \{A - x : A \in \mathcal{H}(x)\}$.*

Proof of Lemma 12: First, we will show that T_x is an $(n-1)$-level complete binary tree by showing that there are $n-1$ nodes in each root-to-leaf path in T_x, there are 2^{n-2} such paths, and each node has zero or two children. Since we delete each subtree in T rooted at a node labeled x' to form T_x, and since x and x' are a heavy sibling pair relative to L, we delete the 2^{n-2} root-to-leaf paths in T corresponding to the 2^{n-2} edges in $\mathcal{H}$ which contain x'. This leaves 2^{n-2} paths, each of which contains a node N which is labeled x. We then contract the tree edge between N and its parent node N^p (and keep the name N^p for the contracted node) to give 2^{n-2} root-to-leaf paths in T_x with length $n-1$.

To show that each node in T_x has zero or two children, we consider an arbitrary node $N \in V(T_x)$. If N was not the parent of a pair of nodes labeled x and x' in T, i.e., not a contracted node, then the children of N are not affected when we prune and contract T. Thus, N retains its zero or two children in T_x. If N was the parent of a pair of nodes N_x and $N_{x'}$ labeled x and x', respectively, in T, i.e., N is a contracted node, then N loses its two children when we prune and contract T, but inherits the zero or two children of N_x. Thus, every node in T_x has zero or two children and we conclude that T_x is a complete $(n-1)$-level binary tree. (See Figure 6.)

Now let $L_x : V(T_x) \to \mathbb{N}$ such that $L_x(N) = L(N)$ for all $N \in V(T_x)$. Note that L_x is well-defined because $V(T_x) \subseteq V(T)$. We will prove that L_x is a good labeling on T_x. Arguments similar to those given above allow us to show that if N and N' are siblings in T_x, then N and N' are siblings in T, and if $N_i < N_j$ in T_x, then $N_i < N_j$ in T. As in the proof of Lemma 3, Properties 1 and 3 hold because if N and N' are siblings in T_x, then they are siblings in T. Property 2 holds because if $N_i < N_j$ in T_x, then $N_i < N_j$ in T. Thus, L_x is a good labeling of T_x.

Let $\mathcal{H}_x$ be the $(n-1)$-uniform tumbleweed realized by L_x on T_x. We will show that $\mathcal{H}_x = \{A - x : A \in \mathcal{H}(x)\}$. In forming T_x, after we delete every subtree rooted at a node labeled x', every root-to-leaf path contains a node labeled x. Then we contract the tree edges between the nodes labeled x and their parent nodes. This reveals a bijection between the root-to-leaf paths in T_x and the root-to-leaf paths which realize the edges in $\mathcal{H}(x)$. The only distinction between the the root-to-leaf paths in T_x and root-to-leaf paths in T which realize the edges in $\mathcal{H}(x)$ is that the root-to-leaf paths in T_x are missing the nodes labeled x. Thus, $\mathcal{H}_x = \{A - x : A \in \mathcal{H}(x)\}$, since $\mathcal{H}_x$ is the hypergraph realized by L_x on T_x. □

Recall that $\mathcal{H}_1 = \{A - r : A \in \mathcal{H}\}$, where r is the root of $\mathcal{H}$.

Lemma 13 *Let $\mathcal{H}$ be an n-uniform tumbleweed. If x and x' are a heavy pair in $\mathcal{H}$, then $\mathcal{H}_1(x) = \{A \in \mathcal{H}_1 : x \in A\}$ is isomorphic to $\mathcal{H}_x = \{A - x : A \in \mathcal{H}(x)\}$, thus, $\mathcal{H}_1(x)$ is an $(n-1)$-uniform tumbleweed.*

Proof of Lemma 13: Let $\mathcal{H}$ be an n-uniform tumbleweed, let x and x' be a heavy pair in $\mathcal{H}$, and let r be the root of $\mathcal{H}$. It is fairly easy to check that $\mathcal{H}_1(x)$ is the same hypergraph as $\mathcal{H}_x$, except the root of $\mathcal{H}_1(x)$ is called x while the root of $\mathcal{H}_x$ is called r. Specifically, both hypergraphs can be obtained from $\mathcal{H}(x) = \{A \in \mathcal{H} : x \in A\}$. To obtain $\mathcal{H}_1(x)$, we delete the vertex r from each edge in $\mathcal{H}(x)$, i.e., $\mathcal{H}_1(x) = \{A - r : A \in \mathcal{H}(x)\}$. To obtain $\mathcal{H}_x$, we delete the vertex x from each edge in $\mathcal{H}(x)$, i.e., $\mathcal{H}_x = \{A - x : A \in \mathcal{H}(x)\}$. Since $\{r, x\} \subseteq A$ for every $A \in \mathcal{H}(x)$, we see that $\mathcal{H}_1(x)$ is isomorphic to $\mathcal{H}_x$. Thus, Lemma 12 implies that $\mathcal{H}_1(x)$ is an $(n-1)$-uniform tumbleweed. □

Theorem 14 *If $\mathcal{H}$ is an n-uniform tumbleweed and L is a labeling of a complete n-level binary tree T which realizes $\mathcal{H}$, then L is a good labeling.*

Proof of Theorem 14: We will prove by induction on n that any labeling of a complete n-level binary tree which realizes an n-uniform tumbleweed is a good labeling. If $\mathcal{H}$ is a 1-uniform tumbleweed, then there is only one labeling of the single node in a 1-level complete binary tree which realizes $\mathcal{H}$, and it is trivially a good labeling. So, assume that any labeling of an $(n-1)$-level complete binary tree which realizes an $(n-1)$-uniform tumbleweed is a good labeling. Let $\mathcal{H}$ be an n-uniform tumbleweed. Let $L : V(T) \to \mathbb{N}$ be an arbitrary labeling on a complete n-level binary tree T which realizes $\mathcal{H}$. Let N_r be the root node of T, and let N and N' be the children of N_r in level 2.

Suppose that $L(N_r) = r, L(N) = x$, and $L(N') = y$. Since $\mathcal{H}$ is a tumbleweed, r must be the root vertex of $\mathcal{H}$. Since L realizes $\mathcal{H}$, and $\mathcal{H}$ is n-uniform, then L satisfies Property 2 of a good labeling. Thus, $x \neq r$ and $y \neq r$. Since there are 2^{n-2} root-to-leaf paths in T which contain N (i.e., the paths that pass through the subtree rooted at N), then $|\mathcal{H}(x)| \geq 2^{n-2}$. Since $x \neq r$ and $|\mathcal{H}(x)| \geq 2^{n-2}$, Lemma 8 implies that $|\mathcal{H}(x)| = 2^{n-2}$. Thus, when we consider N', we know that no node $N_i \geq N'$ is labeled with x, otherwise we would have $|\mathcal{H}(x)| > 2^{n-2}$. We can use similar arguments to conclude that $|\mathcal{H}(y)| = 2^{n-2}$ and no node $N_i \geq N$ is labeled with y. Thus, $\mathcal{H}(x)$ and $\mathcal{H}(y)$ partition $\mathcal{H}$ and $|\mathcal{H}(x)| = |\mathcal{H}(y)| = 2^{n-2}$,

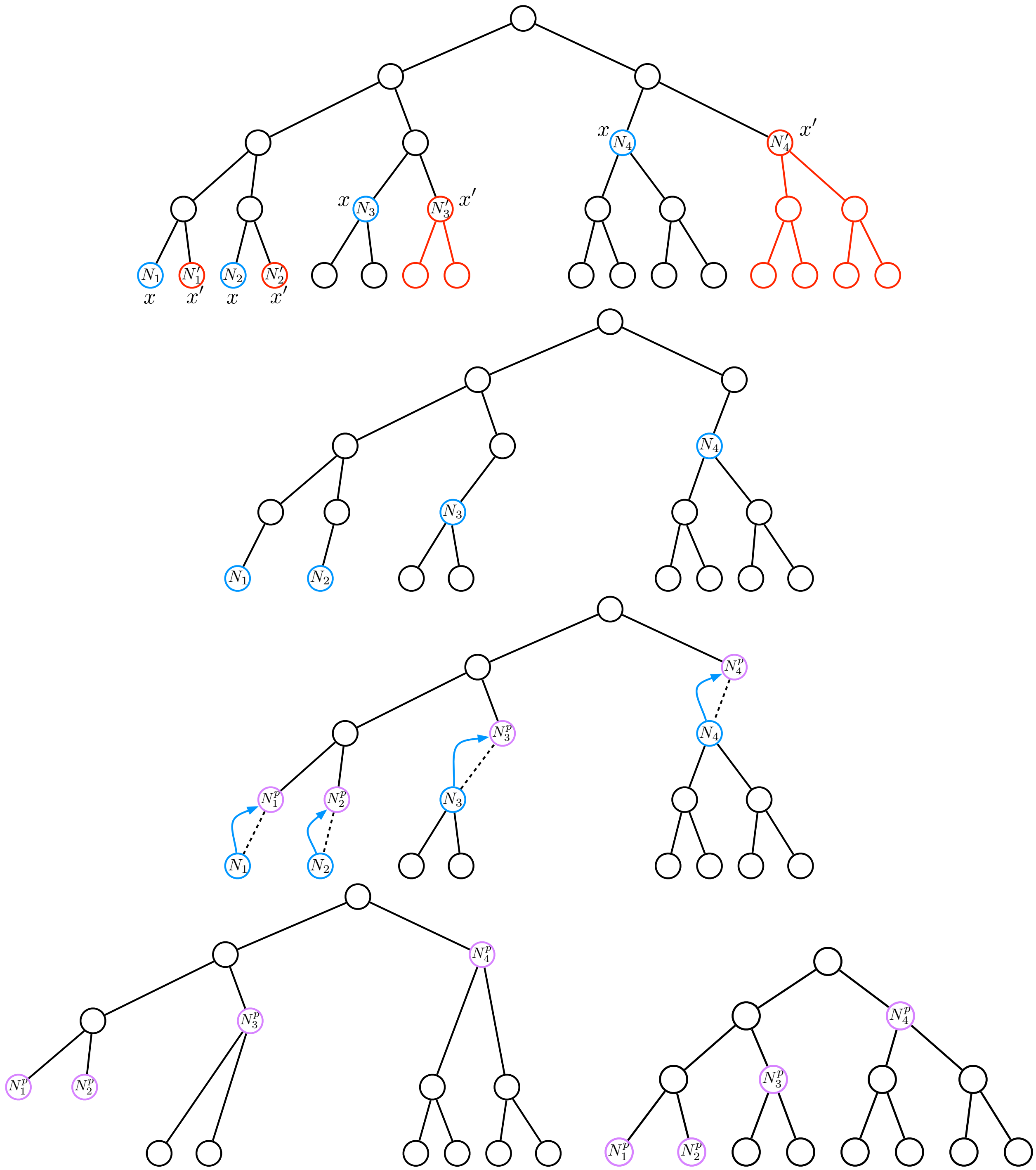

Figure 6: Locate nodes labeled x and x'; delete subtrees rooted at nodes labeled x'; contract tree edges between nodes labeled x and their parents; redraw the resulting complete binary tree.

i.e., x and y form a heavy pair in $\mathcal{H}$. Lemma 9 implies that x and y are a heavy sibling pair relative to every good labeling which realizes $\mathcal{H}$.

Since x and y form a heavy pair in $\mathcal{H}$, Lemma 13 implies that both $\mathcal{H}_1(x)$ and $\mathcal{H}_1(y)$ are $(n-1)$-uniform tumbleweeds. Let T_N and $T_{N'}$ be the subtrees of T rooted at N and N', respectively. It is easy to see that L restricted to T_N yields $\mathcal{H}_1(x)$ and L restricted to $T_{N'}$ yields $\mathcal{H}_1(y)$. Since $\mathcal{H}_1(x)$ and $\mathcal{H}_1(y)$ are $(n-1)$-uniform tumbleweeds, our inductive hypothesis implies that L restricted to T_N and L restricted to $T_{N'}$ are good labelings.

We already know that L satisfies Property 2 of a good labeling. Our inductive hypothesis almost gives us that L also satisfies Property 1. Specifically, since L restricted to T_N and L restricted to $T_{N'}$ are good labelings, any pair of siblings N_i and N_i' contained in T_N or contained in $T_{N'}$ receive different labels in L. The only pair of siblings in T which is not contained in T_N or $T_{N'}$ is the pair of level 2 nodes, N and N', which we already know receive the distinct labels x and y. Thus, L satisfies Property 1 of a good labeling.

To prove that L satisfies Property 3 of a good labeling, we'll show that for any pair of siblings N_i and N_i' in T, if $L(N_i) = a$ and $L(N_i') = b$, then a and b are siblings relative to every good labeling which realizes $\mathcal{H}$. We'll use an inductive argument which shows that an arbitrary pair of siblings N_i and N_i' in level $n-k$ of T satisfies the above property given that every pair of siblings in levels $n-k+1$ through n satisfies the property. It will be convenient to introduce the following definition for use in this proof. If s and s' are siblings in $\mathcal{H}$ relative to every good labeling which yields $\mathcal{H}$, then we say that s and s' are *global siblings* in $\mathcal{H}$.

Let N_i and N_i' be an arbitrary pair of siblings in level $n-k$ of T for some $0 \leq k \leq n-3$, i.e., N_i and N_i' are both in T_N or both in $T_{N'}$. Without loss of generality, assume that N_i and N_i' are both in T_N. Let $L(N_i) = a$ and $L(N_i') = b$. Assume that for any pair of siblings N_j and N_j' in levels $n-k+1$ through n of T, their labels s_j and s_j' relative to L are global siblings in $\mathcal{H}$. Let P_1 be a root-to-leaf path in T which contains N_i, and let A_1 be the corresponding edge of $\mathcal{H}$ realized by labeling L. Let $(v_1, \ldots, v_{n-k-1}, a, s_1, \ldots, s_k)$ be the ordered labels which appear on P_1 so that $v_1 = r$ and $A_1 = \{v_1, \ldots, v_{n-k-1}, a, s_1, \ldots, s_k\}$. Notice that our inductive assumption implies that if N_j is a node in P_1 with $L(N_j) = s_j$ for some $1 \leq j \leq k$, and N_j' is the sibling of N_j, with $L(N_j') = s_j'$, then s_j and s_j' are global siblings in $\mathcal{H}$. Now we construct a root-to-leaf path in T called P_2 which goes through N_i'. At each level between $n-k+1$ and n, we have two choices for the node to include in P_2. If we encounter a pair of siblings N_j and N_j' such that $L(N_j) = s_j \in A_1$ and $L(N_j') = s_j'$, then we select N_j for P_2. Otherwise, we arbitrarily choose one of the siblings for P_2. Let A_2 be the corresponding edge of $\mathcal{H}$ realized by L on P_2. Let $(v_1, \ldots, v_{n-k-1}, b, \sigma_1, \ldots, \sigma_k)$ be the ordered labels which appear on P_2 so that $v_1 = r$ and $A_2 = \{v_1, \ldots, v_{n-k-1}, b, \sigma_1, \ldots, \sigma_k\}$. We also note that our inductive assumption implies that if N_j is a node in P_2 with $L(N_j) = \sigma_j$ for some $1 \leq j \leq k$, and N_j' is the sibling of N_j, with $L(N_j') = \sigma_j'$, then σ_j and σ_j' are global siblings in $\mathcal{H}$. (See Figure 7 for a possible picture of the paths P_1 and P_2 when T is labeled with L.)

For each pair of labels $s_i \in A_1$ and $\sigma_j \in A_2$ such that $s_i = \sigma_j$, rename s_i and σ_j with v_t so that $A_1 = \{v_1, \ldots, v_{n-\ell-1}, a, s_{i_1}, \ldots, s_{i_\ell}\}$ and $A_2 = \{v_1, \ldots, v_{n-\ell-1}, b, \sigma_{j_1}, \ldots, \sigma_{j_\ell}\}$, where $0 \leq \ell \leq k$. (We include $\ell = 0$ for the case when there are no s_i's or σ_j's remaining or there were none to begin with.) For ease of notation, we will drop the double-subscripts and simply write s_t for s_{i_t} and σ_t for σ_{j_t} for each $1 \leq t \leq \ell$ so that $A_1 = \{v_1, \ldots, v_{n-\ell-1}, a, s_1, \ldots, s_\ell\}$ and $A_2 = \{v_1, \ldots, v_{n-\ell-1}, b, \sigma_1, \ldots, \sigma_\ell\}$.

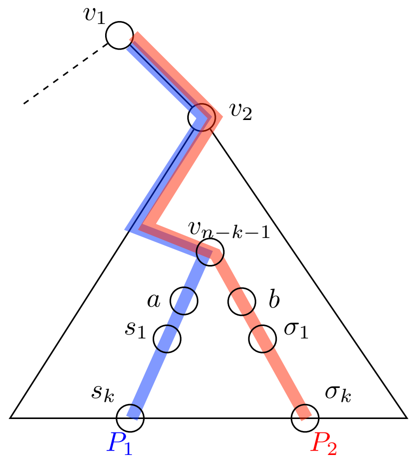

Figure 7: A picture of the the paths P_1 and P_2 when T is labeled with L.

Given these new expressions for A_1 and A_2, we will show that $s_i, s_i' \notin A_2$ and $\sigma_i, \sigma_i' \notin A_1$ for any $1 \leq i \leq \ell$. We proceed by cases. We know that $\{s_i, s_i'\} \cap \{\sigma_j, \sigma_j'\} = \emptyset$ for all $1 \leq i \leq \ell$ and $1 \leq j \leq \ell$, due to how we constructed A_2 and renamed the s_i's and σ_j's which equaled each other. Furthermore, we will show that $a \notin \{\sigma_j, \sigma_j'\}$ and $b \notin \{s_j, s_j'\}$ for all $1 \leq j \leq \ell$. Since N_i and N_i' are nodes in T_N, then their labels a and b are siblings in the tumbleweed $\mathcal{H}_1(x)$ realized by L restricted to T_N, which is a good labeling. If $a = \sigma_j'$, then $b = \sigma_j$ because L restricted to T_N satisfies Property 3 of a good labeling. But then A_2 contains σ_j twice, which is a contradiction. If $a = \sigma_j$, then $b = \sigma_j'$, and $A_2 - r$ contains both σ_j and σ_j' which contradicts Lemma 4 when applied to $\mathcal{H}_1(x)$. Thus, $a \notin \{\sigma_j, \sigma_j'\}$ for all $1 \leq j \leq \ell$. Similar arguments show that $b \notin \{s_j, s_j'\}$ for all $1 \leq j \leq \ell$. Clearly, $s_i \neq v_j$ and $\sigma_i \neq v_j$ for any i and j. If $v_j = s_i'$ for some i and j, then $s_i, s_i' \in A_1$ which contradicts Lemma 4 applied to $\mathcal{H}$, since s_i and s_i' are global siblings. Thus, $v_j \neq s_i'$ for any i and j. A similar argument gives us that $v_j \neq \sigma_i'$ for any i and j. This allows us to conclude that $s_i, s_i' \notin A_2$ and $\sigma_i, \sigma_i' \notin A_1$ for any $1 \leq i \leq \ell$.

Let L_g be an arbitrary good labeling which realizes $\mathcal{H}$. We know that there must be root-to-leaf paths P_1' and P_2' in T which correspond to A_1 and A_2, respectively, when T is labeled with L_g. We will show by finding the ordered labels on P_1' and P_2' that a and b are siblings in $\mathcal{H}$ relative to L_g.

First, we will consider the ordering of the labels on P_1' relative to L_g. Since $v_1 = r$ is the root of $\mathcal{H}$, Lemma 7 implies that the root N_r of T must be labeled with v_1. So, we know that the ordered labels on P_1' relative to L_g in the first t levels of T are $(v_1, v_{i_2}, \ldots, v_{i_t})$ for some $1 \leq t \leq n-\ell-1$, where the v_{i_j}'s are vertices in $A_1 \cap A_2$. We will show that the first t nodes in P_2' also receive the labels $(v_1, v_{i_2}, \ldots, v_{i_t})$ relative to L_g. To do so, we will assume towards a contradiction that at some level j in T where $1 \leq j \leq t$, P_2' receives the label v_{i_j}' instead of v_{i_j}, where v_{i_j} and v_{i_j}' are siblings in $\mathcal{H}$ relative to L_g. But since $v_{i_j} \in A_1 \cap A_2$, then we have $v_{i_j} \in A_2$ and $v_{i_j}' \in A_2$, which contradicts Lemma 4. Therefore, P_1' and P_2' receive the same labels $(v_1, v_{i_2}, \ldots, v_{i_t})$ in the first t levels of T.

Now let t be the largest index such that the initial portion of ordered labels of P_1' relative to L_g up to level t consists solely of v_{i_j}'s, i.e., the ordered labeling of P_1' relative to L_g is $(v_1, v_{i_2}, \ldots, v_{i_t}, w_1, \ldots, w_{n-t})$ and $w_1 \notin A_1 \cap A_2$, thus, $w_1 = a$ or $w_1 = s_i$ for some $1 \le i \le \ell$. Assume towards a contradiction that $w_1 = s_i$ for some $1 \le i \le \ell$. From above, we know that P_2' has the same initial portion of ordered labels $(v_1, v_{i_2}, \ldots, v_{i_t})$. Since s_i and s_i' are global siblings, they are the only available choices for the level $t+1$ label of P_2' relative to L_g. But this forces either $s_i \in A_2$ or $s_i' \in A_2$, which is a contradiction, since we previously established that $s_i, s_i' \notin A_2$. Thus, we may assume that $w_1 = a$. Let w_1' be the sibling of w_1 relative to L_g. Therefore, the only available choices for the level $t+1$ label of P_2' relative to L_g are w_1 and w_1'. We know that $a \neq b$ because a and b are siblings in $\mathcal{H}_1(x)$ and we know that $a \neq v_j$ and $a \neq \sigma_j$ for all j, thus, $w_1 \notin A_2$. This forces the initial portion of labels of P_2' to be $(v_1, v_{i_2}, \ldots, v_{i_t}, w_1')$ and $w_1' \in A_2$. If $w_1' = v_j$ for some j, then a and v_j are siblings relative to L_g, yet $a \in A_1$ and $v_j \in A_1$ which contradicts Lemma 4. If $w_1' = \sigma_j$ for some j, then because σ_j and σ_j' are global siblings, we have $a = \sigma_j'$ which contradicts what we established above. Thus, $w_1' = b$. This establishes that a and b are siblings relative to L_g. Since L_g is an arbitrary good labeling which realizes $\mathcal{H}$, then a and b are global siblings in $\mathcal{H}$. Therefore, by induction, if N_i and N_i' are siblings in level j of T, where $3 \le j \le n$, and $L(N_i) = a$ and $L(N_i') = b$, then a and b are global siblings in $\mathcal{H}$. Since x and y are a heavy sibling pair relative to every good labeling which realizes $\mathcal{H}$, then x and y are also global siblings in $\mathcal{H}$.

To finish establishing that L satisfies Property 3, suppose that N_i and N_i' are siblings in T and N_j and N_j' are siblings in T, and $L(N_i) = L(N_j) = a$ and $L(N_i') = b$ and $L(N_j') = c$. We established above that a and b are global siblings in $\mathcal{H}$, thus, in every good labeling which realizes $\mathcal{H}$, a and b must be siblings. However, the same is true for a and c. The only way this can be accomplished is if $b = c$, i.e., if L satisfies Property 3. Thus, L is a good labeling of T. Therefore by induction, every labeling which realizes $\mathcal{H}$ must be a good labeling. Moreover, our proof reveals that if a and b are siblings relative to some labeling which realizes $\mathcal{H}$, then a and b are siblings relative to every labeling which realizes $\mathcal{H}$. We restate this fact in the following corollary. $\square$

Corollary 15 *Let $\mathcal{H}$ be an n-uniform tumbleweed and let L be a labeling on a complete n-level binary tree T which realizes $\mathcal{H}$. If N and N' are siblings in T and $L(N) = a$ and $L(N') = b$, then a and b are siblings relative to every good labeling which realizes $\mathcal{H}$.*

Thus, Corollary 15 allows us to say that if x and x' are siblings relative to some labeling which realizes a tumbleweed $\mathcal{H}$, then x and x' are *siblings* in $\mathcal{H}$.

4 Economical Extremal Hypergraphs

In this section, we give two distinct characterizations of the economical extremal hypergraphs for the Erdős–Selfridge theorem in Theorem 18 and Corollary 25. *Throughout this section we will assume that every Maker's move we consider is made according to her economical winning strategy.* Let $\mathcal{H}$ be an n-uniform economical extremal hypergraph for the Erdős–Selfridge theorem. We know from Lemma 11 that Maker's first move must be the root r of $\mathcal{H}$, thus, $\mathcal{H}_1 = \{A - r : A \in \mathcal{H}\}$ is the set of partial edges after Maker's first move. For a hypergraph $\mathcal{F}$, let $\mathcal{F}(x, \overline{y}) = \{A \in \mathcal{F} : x \in A, y \notin A\}$, i.e., the set of edges of $\mathcal{F}$ which contain x but do not contain y.

Lemma 16 *If $\mathcal{H}$ is an n-uniform economical extremal hypergraph and y_1 is Breaker's first move and x_2 is Maker's response, then $\mathcal{H}_1(x_2, \overline{y_1})$ is an $(n-1)$-uniform economical extremal hypergraph. Moreover, $\mathcal{H}_1(x_2, \overline{y_1}) = \mathcal{H}_1(x_2)$.*

Proof of Lemma 16: We will consider two potential moves by Breaker in his first turn and the responses made by Maker. First, suppose that Breaker takes y_1 as his first move, then Maker responds with x_2 as her second move. Let $\mathcal{H}_1^{(1)} = \mathcal{H}_1(x_2, \overline{y_1})$, the set of surviving partial edges after Breaker's first move y_1 which contain Maker's second move x_2. Suppose that instead Breaker takes $y_1^{(2)} = x_2$ as his first move, then Maker responds with $x_2^{(2)}$. Let $\mathcal{H}_1^{(2)} = \mathcal{H}_1\left(x_2^{(2)}, \overline{x_2}\right)$, the set of surviving partial edges after Breaker's new first move x_2 which contain Maker's response $x_2^{(2)}$.

When Breaker chooses y_1 and Maker responds with x_2, then Maker can win the game on $\mathcal{H}$ in n turns only if she has an $(n-1)$-turn winning strategy for the Maker–Breaker game played on $\mathcal{H}_1^{(1)}$. Since Maker has an n-turn winning strategy on $\mathcal{H}$, the Erdős–Selfridge theorem implies that $|\mathcal{H}_1^{(1)}| \geq 2^{n-2}$. The same argument applied to $\mathcal{H}_1^{(2)}$ when Breaker chooses $y_1^{(2)} = x_2$ and Maker responds with $x_2^{(2)}$ allows us to conclude that $|\mathcal{H}_1^{(2)}| \geq 2^{n-2}$.

Since every edge in $\mathcal{H}_1^{(1)}$ contains x_2 and every edge in $\mathcal{H}_1^{(2)}$ does not contain x_2, we know that $\mathcal{H}_1^{(1)} \cap \mathcal{H}_1^{(2)} = \emptyset$. Thus, $|\mathcal{H}_1^{(1)} \cup \mathcal{H}_1^{(2)}| = |\mathcal{H}_1^{(1)}| + |\mathcal{H}_1^{(2)}| \geq 2(2^{n-2})$. However, $\mathcal{H}_1^{(1)} \cup \mathcal{H}_1^{(2)} \subseteq \mathcal{H}_1$ and $|\mathcal{H}_1| = 2^{n-1}$, which leads us to conclude that $|\mathcal{H}_1^{(1)}| = 2^{n-2}$, $|\mathcal{H}_1^{(2)}| = 2^{n-2}$, and $\mathcal{H}_1^{(1)}$ and $\mathcal{H}_1^{(2)}$ partition $\mathcal{H}_1$.

Since Maker has an $(n-1)$-turn winning strategy on $\mathcal{H}_1^{(1)} = \mathcal{H}_1(x_2, \overline{y_1})$ and $|\mathcal{H}_1^{(1)}| = 2^{n-2}$, then $\mathcal{H}_1^{(1)}$ is an $(n-1)$-uniform economical extremal hypergraph for the Erdős–Selfridge theorem.

To see that $\mathcal{H}_1(x_2, \overline{y_1}) = \mathcal{H}_1(x_2)$, note that $\mathcal{H}_1(x_2, \overline{y_1}) \subseteq \mathcal{H}_1(x_2)$. On the other hand, if $A \in \mathcal{H}_1(x_2)$, then $A \notin \mathcal{H}_1(x_2^{(2)}, \overline{x_2})$. Thus, $A \in \mathcal{H}_1(x_2, \overline{y_1})$, because $\mathcal{H}_1^{(1)}$ and $\mathcal{H}_1^{(2)}$ partition $\mathcal{H}_1$. Therefore, $\mathcal{H}_1(x_2) \subseteq \mathcal{H}_1(x_2, \overline{y_1})$. □

Recall that we say a tumbleweed $\mathcal{H}$ has the *Single Label Property* (SLP) if for each vertex $v \in V(\mathcal{H})$, there exists a good labeling of a complete binary tree T which realizes $\mathcal{H}$ in which v appears as a label exactly once. We will prove that a hypergraph $\mathcal{H}$ is an n-uniform economical extremal hypergraph if and only if $\mathcal{H}$ is an n-uniform tumbleweed with the SLP.

Corollary 17 (Corollary to Lemma 12) *If $\mathcal{H}$ is an n-uniform tumbleweed with the Single Label Property and x and x' are a heavy pair in $\mathcal{H}$, then $\mathcal{H}_x$ is an $(n-1)$-uniform tumbleweed with the Single Label Property.*

Proof of Corollary 17: Let $\mathcal{H}$ be an n-uniform tumbleweed with the Single Label Property. Let x and x' be a heavy pair in $\mathcal{H}$. Consider an arbitrary vertex $u \in V(\mathcal{H}_x)$. We know that $u \in V(\mathcal{H})$ as well, and since $\mathcal{H}$ has the Single Label Property, there is a good labeling L of T which realizes $\mathcal{H}$ in which u appears as a label exactly once. Recall that we prune and contract T to give T_x by deleting all subtrees rooted at nodes labeled x' and contracting the tree edges between each node N labeled x and its parent node N^p, and keep the name N^p for the contracted node. Recall that $L_x : V(T_x) \to \mathbb{N}$ is the labeling of $V(T_x)$ such that $L_x(N) = L(N)$ for all $N \in V(T_x)$. Then by Lemma 12, $\mathcal{H}_x$ is an $(n-1)$-uniform tumbleweed, and L_x is a good labeling on T_x which realizes $\mathcal{H}_x$. Then since u appears exactly once in L and since $u \in V(\mathcal{H}_x)$, u appears exactly once in L_x as well. As u is an arbtirary vertex in $\mathcal{H}_x$, such a good labeling can be found for any vertex in $\mathcal{H}_x$. Therefore, $\mathcal{H}_x$ is an $(n-1)$-uniform tumbleweed with the Single Label Property. $\square$

Theorem 18 *An n-uniform hypergraph $\mathcal{H}$ is an economical extremal hypergraph for the Erdős–Selfridge theorem if and only if $\mathcal{H}$ is a tumbleweed with the Single Label Property.*

Proof of Theorem 18: First, we will prove by induction on n that every n-uniform EEH for the Erdős-Selfridge theorem is an n-uniform tumbleweed with the SLP.

Let $\mathcal{H}$ be an n-uniform EEH. When $n = 1$, $\mathcal{H}$ consists of a single edge with exactly 1 vertex, which is trivially a tumbleweed with the SLP.

Now suppose $n \geq 2$, and assume that every k-uniform EEH for the Erdős-Selfridge theorem with $k < n$ is a k-uniform tumbleweed with the SLP. Let $\mathcal{H}_1^{(1)}$ and $\mathcal{H}_1^{(2)}$ be defined as in the proof of Lemma 16. By Lemma 16, we know that $\mathcal{H}_1^{(i)}$ is an $(n-1)$-uniform EEH for $i \in \{1, 2\}$. So, by the inductive hypothesis, $\mathcal{H}_1^{(i)}$ is an $(n-1)$-uniform tumbleweed with the SLP for $i \in \{1, 2\}$. Let $L^{(i)}$ be a good labeling on a complete $(n-1)$-level binary tree $T^{(i)}$ which realizes $\mathcal{H}_1^{(i)}$ for each $i \in \{1, 2\}$.

Now let T be the complete n-level binary tree whose root node is N_r and whose left and right subtrees are $T^{(1)}$ and $T^{(2)}$, respectively. Let $L : V(T) \to \mathbb{N}$ be the labeling of T in which $L(N_r) = x_1$ and $L(N) = L^{(i)}(N)$ for all $N \in V(T^{(i)})$ for $i \in \{1, 2\}$. Notice that the labeling L of T realizes $\mathcal{H}$. Specifically, $A \in \mathcal{H}$ if and only if $A - x_1 \in \mathcal{H}_1$ if and only if $A - x_1 \in \mathcal{H}_1^{(i)}$ for $i = 1$ or $i = 2$ (because $\mathcal{H}_1^{(1)}$ and $\mathcal{H}_1^{(2)}$ partition $\mathcal{H}_1$) if and only if there is a root-to-leaf path in $T^{(i)}$ which realizes $A - x_1$ when $T^{(i)}$ is labeled with $L^{(i)}$ for $i = 1$ or $i = 2$ if and only if there is a root-to-leaf path in T which realizes A when T is labeled with L.

Next, we show that L is a good labeling of T, and hence, $\mathcal{H}$ is a tumbleweed. Our arguments are similar to those in the proof of Theorem 14. For example, since L realizes $\mathcal{H}$, and $\mathcal{H}$ is n-uniform, then L satisfies Property 2 of a good labeling. To show that L satisfies Property 1, suppose that N and N' are siblings in T. If N and N' are both in $T^{(i)}$ for $i = 1$ or $i = 2$, then $L(N) \neq L(N')$ since $L^{(i)}$ is a good labeling for $i \in \{1, 2\}$. The only remaining siblings in T are the roots of $T^{(1)}$ and $T^{(2)}$, which are labeled with x_2 and $x_2^{(2)}$, respectively. Since $\mathcal{H}_1^{(2)} = \mathcal{H}_1(x_2^{(2)}, \overline{x_2}) \neq \emptyset$, then $x_2 \neq x_2^{(2)}$. Thus, L satisfies Property 1.

Finally, we must show that L satisfies Property 3. Let N_j and N_j' be siblings in T and let N_ℓ and N_ℓ' also be siblings in T. Suppose that $L(N_j) = L(N_\ell)$. We know that if $N_j, N_j', N_\ell, N_\ell' \in V(T^{(i)})$ for $i = 1$ or $i = 2$, then $L(N_j') = L(N_\ell')$, since $L^{(i)}$ is a good labeling for $i \in \{1, 2\}$.

So we may assume that $N_j, N_j' \in V(T^{(1)})$ and $N_\ell, N_\ell' \in V(T^{(2)})$. (Other cases are either trivial, such as, when $N_j = N_\ell$, or never occur, such as, when N_j is the root of $T^{(1)}$ and $N_\ell \in V(T^{(2)})$.) Let us also assume that $L(N_j) = L(N_\ell) = u$, $L(N_j') = w$, and $L(N_\ell') = z$. We consider another set of alternative Breaker moves and Maker responses in order to gain more insight into the structure of $\mathcal{H}$. Specifically, consider the case where Breaker occupies u

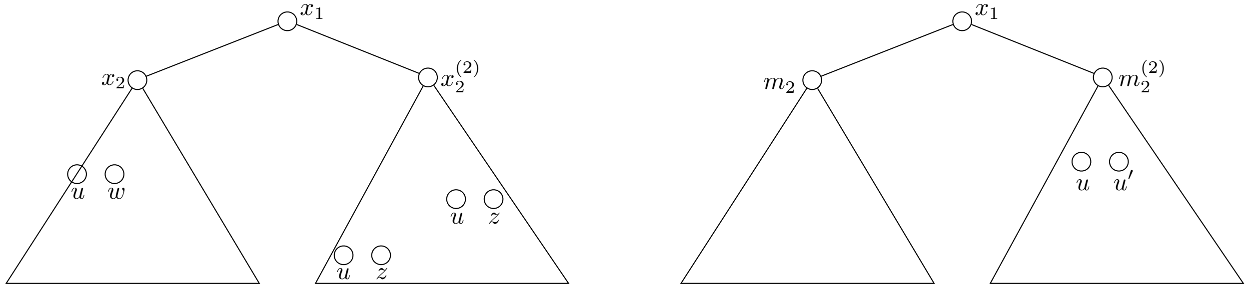

Figure 8: On the left is the labeling L which realizes $\mathcal{H}$, and on the right is the labeling L' which also realizes $\mathcal{H}$, but uses label u exactly once.

for his first move and Maker responds with m_2. Let $\mathcal{H}_1^{(1')} = \mathcal{H}_1(m_2, \overline{u})$ be the set of surviving partial edges after Breaker's first move u which contain Maker's response m_2. Let us also consider the case where instead Breaker occupies m_2 for his first move and Maker responds with $m_2^{(2)}$. Let $\mathcal{H}_1^{(2')} = \mathcal{H}_1(m_2^{(2)}, \overline{m_2})$ be the set of surviving partial edges after Breaker's first move m_2 which contain Maker's response $m_2^{(2)}$. Again, by Lemma 16, we know that $\mathcal{H}_1^{(i')}$ is an $(n-1)$-uniform economical extremal hypergraph for $i \in \{1,2\}$. So, by the inductive hypothesis, $\mathcal{H}_1^{(i')}$ is an $(n-1)$-uniform tumbleweed with the SLP for $i \in \{1,2\}$. Furthermore, by construction, $u \notin V(\mathcal{H}_1^{(1')})$, thus, $u \in V(\mathcal{H}_1^{(2')})$. Therefore u does not appear in any good labeling which realizes $\mathcal{H}_1^{(1')}$; and since $\mathcal{H}_1^{(2')}$ has the SLP, there exists a good labeling which realizes $\mathcal{H}_1^{(2')}$ in which u appears exactly once. Let $L^{(1')}$ be a good labeling of $T^{(1)}$ which realizes $\mathcal{H}_1^{(1')}$, and let $L^{(2')}$ be a good labeling of $T^{(2)}$ which realizes $\mathcal{H}_1^{(2')}$ in which u appears exactly once. Then let L' be the labeling of T in which $L'(N_r) = x_1$ and $L'(N) = L^{(i')}(N)$ if $N \in V(T^{(i)})$ for $i \in \{1,2\}$. Clearly, the label u appears exactly once in this labeling L', so u has a single sibling, u', in this labeling. Using the same arguments that we used to prove that the labeling L of T realizes $\mathcal{H}$, we can show that the labeling L' of T which contains u exactly once also realizes $\mathcal{H}$. (See Figure 8 for a possible picture of the the labelings L and L'.)

Similar to the proof of Theorem 14, we will examine root-to-leaf paths in T which pass through N_j and N_j' and use L to realize edges $A_1, A_2 \in \mathcal{H}$ (so that $L(N_j) = u \in A_1$ and $L(N_j') = w \in A_2$). Then we use L' to label T and search for the root-to-leaf paths which realize A_1 and A_2. Since u appears as a label exactly once when using L', we are able to prove that L has Property 3 of a good labeling.

Suppose that the sibling pair N_j and ${N_j}'$ is in level k of T. Let P_1 be a root-to-leaf path in T which contains N_j, and let A_1 be the corresponding edge of $\mathcal{H}$ realized by labeling L. Let $(v_1, \dots, v_{k-1}, u, s_1, \dots, s_{n-k})$ be the ordered labels which appear on P_1 so that $v_1 = x_1$ and $A_1 = \{v_1, \dots, v_{k-1}, u, s_1, \dots, s_{n-k}\}$. Now we construct a root-to-leaf path in T called P_2 which goes through N_j'. At each level between $k+1$ and n, we have two choices for the node to include in P_2. If we encounter a pair of siblings N_t and N_t' such that $L(N_t) = s_t \in A_1$ and $L(N_t') = s_t'$, then we select N_t for P_2. Otherwise, we arbitrarily choose one of the siblings for P_2. Let A_2 be the corresponding edge of $\mathcal{H}$ realized by L. Let $(v_1, \dots, v_{k-1}, w, \sigma_1, \dots, \sigma_{n-k})$ be the ordered labels which appear on P_2 so that $v_1 = x_1$ and

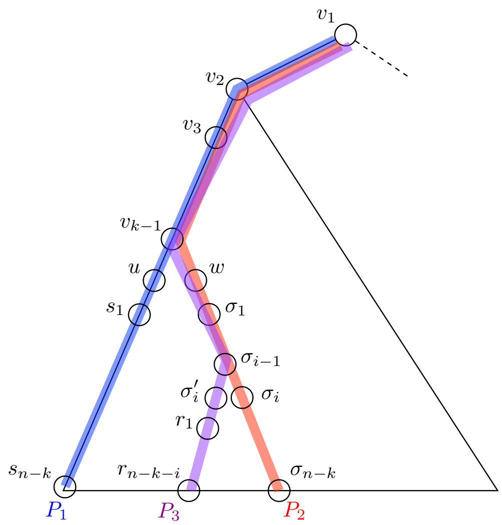

Figure 9: A picture of the the paths P_1 and P_2 and P_3 when T is labeled with L.

$A_2 = \{v_1, \ldots, v_{k-1}, w, \sigma_1, \ldots, \sigma_{n-k}\}$. (See Figure 9 for a possible picture of paths P_1 and P_2 (and P_3, which we define later) when T is labeled with L.) Let P_1' and P_2' be paths in T which realize A_1 and A_2, respectively, when T is labeled with L'. Examining P_1' and P_2' (and other paths) will help us show that u', the sibling of u relative to L', equals w. Repeating the process with N_ℓ and N_ℓ' will allow us to show that u' also equals z, thus allowing us to conclude that $w = z$.

First, we will consider the ordering of the labels on P_1' relative to L'. Notice that L' labels the root of $T^{(1)}$ with m_2 and the root of $T^{(2)}$ with $m_2^{(2)}$. We know that the ordered labels on P_1' must begin with $v_1 = x_1$. Since $u \in A_1$ and $u \notin V(\mathcal{H}_1^{(1')})$, then we know that the root of $T^{(2)}$ must be the second node in P_1', thus $m_2^{(2)} \in A_1$.

Consider the case where $u = m_2^{(2)}$. Notice this implies that $u' = m_2$. We know the first two labels on P_1' will be v_1 and u because $u \in A_1$. We also know that P_2' must have v_1 as its first label. However, by applying Lemma 4 to u, w and $\mathcal{H}_1^{(1)}$, we can conclude that $u \notin A_2 - v_1$. Thus, no node in P_2' receives u as a label, and u' must be the label of the second node in P_2'. Thus, $u' \in \{v_2, \ldots, v_{k-1}, w, \sigma_1, \ldots, \sigma_{n-k}\}$. We will show that $u' = w$. First, assume towards a contradiction that $u' = v_i$ for some $2 \leq i \leq k-1$. In this case, since $u' = m_2$ and $v_i \in A_1 - v_1$, we have $m_2 \in A_1 - v_1$. However, because P_1' passes through $T^{(2)}$, then $A_1 - v_1 \in \mathcal{H}_1^{(2')} = \mathcal{H}(m_2^{(2)}, \overline{m_2})$, thus, $m_2 \notin A_1 - v_1$, which is a clear contradiction. Therefore, $u' \neq v_i$ for $2 \leq i \leq k-1$. Now assume towards a contradiction that $u' = \sigma_i$ for some $1 \leq i \leq n-k$. Let P_3 be a root-to-leaf path in T which passes through N_j' and has the ordered labels $(v_1, v_2, \ldots, v_{k-1}, w, \sigma_1, \ldots, \sigma_{i-1}, {\sigma_i}', r_1, \ldots, r_{n-k-i})$ relative to L where σ_i and σ_i' are siblings in $\mathcal{H}_1^{(1)}$; and let $A_3 = \{v_1, v_2, \ldots, v_{k-1}, w, \sigma_1, \ldots, \sigma_{i-1}, {\sigma_i}', r_1, \ldots, r_{n-k-i}\}$ be the corresponding edge of $\mathcal{H}$ realized by L. (See Figure 9 for a possible picture of path P_3 when T is labeled with L.) We know that A_3 must also correspond to some root-to-leaf path P_3' in T labeled with L', but we will show that no such path exists. Clearly, the label that L' assigns to the first node of any path must be v_1. The two choices for the label that L'

assigns to the second node of any path are u and $u' = \sigma_i$. However, as we did above, we can apply Lemma 4 to $u, w, \sigma_i, \sigma_i'$ and $\mathcal{H}_1^{(1)}$ to conclude that $u \notin A_3 - v_1$ and $\sigma_i \notin A_3 - v_1$. Thus, there is no path P_3' which realizes A_3 relative to L', which is a contradiction. Therefore, it must be the case that $u' = w$ when $u = m_2^{(2)}$.

Now consider the case where $u \neq m_2^{(2)}$. Suppose that the ordered labels assigned by L' to the first t nodes of P_1' are $(v_1, v_{i_2}, \ldots, v_{i_t})$, where $v_{i_2}, \ldots, v_{i_t} \in \{v_2, \ldots, v_{k-1}\}$ with $1 \leq t \leq k-1$, and $v_1 = x_1$. (When $t = 1$, we are only considering the first node of P_1' which is labeled v_1.) If P_j' is a root-to-leaf path which realizes an edge A_j such that $\{v_1, \ldots, v_{k-1}\} \subseteq A_j$, when T is labeled with L', then we will show that P_j' has the same ordered labels on its first t nodes as P_1'. Clearly, the first node of P_j' is labeled v_1. When $t \geq 2$, then $v_{i_2} = m_2^{(2)}$. Since $v_{i_2} \in \{v_2, \ldots, v_{k-1}\} \subseteq A_j - v_1$, then $m_2^{(2)} \in A_j - v_1$. Thus, $A_j - v_1 \in \mathcal{H}_1(m_2^{(2)})$. Lemma 16 implies that $\mathcal{H}_1(m_2^{(2)}) = \mathcal{H}_1^{(2')}$, which means P_j' passes through $T^{(2)}$ and the second node of P_j' is labeled v_{i_2}. Now, assume towards a contradiction that the level c node of P_j' is labeled ${v_{i_c}}'$ instead of v_{i_c}, where v_{i_c} and ${v_{i_c}}'$ are siblings relative to $L^{(2')}$ and $3 \leq c \leq t$. But then $v_{i_c} \in A_j - v_1$ and ${v_{i_c}}' \in A_j - v_1$, which is a contradiction to Lemma 4 applied to $\mathcal{H}_1^{(2')}$. Therefore, if the ordered labels assigned by L' to the first t nodes of P_1' are $(v_1, v_{i_2}, \ldots, v_{i_t})$, then the same is true for P_j'.

Now let t be the largest index such that the initial portion of ordered labels of P_1' relative to L' up to level t (where $1 \leq t \leq k-1$) consists solely of v_j's, i.e., the ordered labeling of P_1' relative to L' is $(v_1, v_{i_2}, \ldots, v_{i_t}, \alpha_1, \ldots, \alpha_{n-t})$ and $\alpha_1 \neq v_j$ (for $1 \leq j \leq k-1$), thus, $\alpha_1 = u$ or $\alpha_1 = s_i$ for some $1 \leq i \leq n-k$. Assume towards a contradiction that $\alpha_1 = s_i$ for some $1 \leq i \leq n-k$. Let P_4 be a root-to-leaf path in T which passes through N_j and has the ordered labels $(v_1, v_2, \ldots, v_{k-1}, u, s_1, \ldots, s_{i-1}, s_i', \beta_1, \ldots, \beta_{n-k-i})$ relative to L where s_i and s_i' are siblings in $\mathcal{H}_1^{(1)}$; and let $A_4 = \{v_1, v_2, \ldots, v_{k-1}, u, s_1, \ldots, s_{i-1}, s_i', \beta_1, \ldots, \beta_{n-k-i}\}$ be the corresponding edge of $\mathcal{H}$ realized by L. Let P_4' be the root-to-leaf path in T which realizes A_4 relative to L'. Since there is a unique node in T labeled u and $u \in A_4$, the initial portion of ordered labels of P_4' must be $(v_1, v_{i_2}, \ldots, v_{i_t}, \alpha_1, \ldots, \alpha_c)$, where $\alpha_c = u$ and $2 \leq c \leq n-t$. Since $\alpha_1 = s_i$, then $s_i \in A_4 - v_1$. But this is a contradiction to Lemma 4 applied to $\mathcal{H}_1^{(1)}$ since $s_i' \in A_4 - v_1$. Thus, we may assume that $\alpha_1 = u$. From above, we know that P_2' has the same initial portion of ordered labels $(v_1, v_{i_2}, \ldots, v_{i_t})$ since $\{v_1, \ldots, v_{k-1}\} \subseteq A_2$. Recall that applying Lemma 4 to u, w and $\mathcal{H}_1^{(1)}$ implies that $u \notin A_2 - v_1$, thus, the level $t+1$ node of P_2' is labeled u'. We will show that $u' = w$. If $u' = v_i$ for some $2 \leq i \leq k-1$, then u and v_i are siblings in $\mathcal{H}_1^{(2')}$. But $u, v_i \in A_1 - v_1 \in \mathcal{H}_1^{(2')}$ which contradicts Lemma 4. If $u' = \sigma_i$ for some $1 \leq i \leq n-k$, then we can show that there is no path P_3' which realizes the edge $A_3 \in \mathcal{H}$ relative to L', (where A_3 is defined as above, namely, $A_3 = \{v_1, v_2, \ldots, v_{k-1}, w, \sigma_1, \ldots, \sigma_{i-1}, {\sigma_i}', r_1, \ldots, r_{n-k-i}\}$). Assume towards a contradiction that P_3' realizes A_3 relative to L'. Since $\{v_1, \ldots, v_{k-1}\} \subseteq A_3$, then $(v_1, v_{i_2}, \ldots, v_{i_t})$ is the initial portion of ordered labels of P_3'. The two choices for the label that L' assigns to the $(t+1)^{th}$ node of P_3' are u and $u' = \sigma_i$. However, as above, we can apply Lemma 4 to $u, w, \sigma_i, \sigma_i'$ and $\mathcal{H}_1^{(1)}$ to conclude that $u \notin A_3 - v_1$ and $\sigma_i \notin A_3 - v_1$. Thus, there is no path P_3' which realizes A_3 relative to L', which is a contradiction. Therefore, it must be the case that $u' = w$ when $u \neq m_2^{(2)}$.

In this way, we have shown that $u' = w$ is the unique sibling of u when T is labeled with

L'. However, we can use the same path-finding arguments with a pair of root-to-leaf paths in T containing N_ℓ and N_ℓ' to prove that $u' = z$ as well, since $L(N_\ell) = u$ and $L(N_\ell{}') = z$. Therefore, we reach the conclusion that $w = z$. So, if N_j and $N_j{}'$ are siblings and N_ℓ and $N_\ell{}'$ are the siblings in T where $L(N_j) = L(N_\ell)$, then $L(N_j{}') = L(N_\ell{}')$. Thus, L also satisfies Property 3; therefore, L is a good labeling on the complete n-level binary tree T which realizes $\mathcal{H}$, so $\mathcal{H}$ is an n-uniform tumbleweed.

Note that since $\mathcal{H}$ is a tumbleweed which is realized by the labeling L' on the binary tree T, according to Theorem 14, L' is a good labeling. And since u was an arbitrary vertex in $V(\mathcal{H})$, such a good labeling can be found for any vertex in $V(\mathcal{H})$. That is, given any vertex $u \in V(\mathcal{H})$, there is a good labeling which realizes $\mathcal{H}$ in which u appears as a label exactly once. So, $\mathcal{H}$ has the SLP. Therefore, by induction, every n-uniform economical extremal hypergraph is an n-uniform tumbleweed with the Single Label Property.

Next, we prove by induction on n that every n-uniform tumbleweed $\mathcal{H}$ with the Single Label Property is an n-uniform economical extremal hypergraph. If $n = 1$, then $\mathcal{H}$ is the hypergraph which can be represented by a labeling of the single node in the 1-level complete binary tree. Clearly, Maker has a 1-turn winning strategy on the game played on $\mathcal{H}$, as Maker will simply occupy this lone vertex in her first move and win the game. Since $|\mathcal{H}| = 1 = 2^{1-1}$, then $\mathcal{H}$ is an economical extremal hypergraph.

Let $n \geq 2$, and assume that every k-uniform tumbleweed with the Single Label Property where $k < n$ is a k-uniform economical extremal hypergraph. We will show that if $\mathcal{H}$ is an n-uniform tumbleweed with the Single Label Property, then $\mathcal{H}$ is an n-uniform economical extremal hypergraph. By Lemmas 1 and 2, $\mathcal{H}$ is an n-uniform hypergraph with 2^{n-1} edges, so we must prove that Maker has an n-turn winning strategy for the game played on $\mathcal{H}$. By Lemma 11, Maker's first move x_1 is the root of $\mathcal{H}$. We note that Maker has an n-turn winning strategy on $\mathcal{H}$, if and only if for every Breaker's first move y_1, there is a vertex x such that Maker has an $(n-1)$-turn winning strategy on $\mathcal{H}_1(x, \overline{y_1})$. Let y_1 be an arbitrary Breaker's first move. Let L be a good labeling of the complete n-level binary tree T in which y_1 appears exactly once as a label. (We know that L exists because $\mathcal{H}$ has the SLP.) Let N and N' be the nodes in level 2 of T. Without loss of generality, the unique node in T which is labeled y_1 is contained in the subtree of T rooted at N'. Let $x = L(N)$ and $x' = L(N')$, so that $y_1 \in V(\mathcal{H}_1(x'))$ and $y_1 \notin V(\mathcal{H}_1(x))$. Since $y_1 \notin V(\mathcal{H}_1(x))$, then $\mathcal{H}_1(x, \overline{y_1}) = \mathcal{H}_1(x)$. Moreover, since x and x' are the labels of the nodes in level 2 of T, we know that x and x' are a heavy pair in $\mathcal{H}$. Lemma 13 and Corollary 17 imply that $\mathcal{H}_1(x)$ is an $(n-1)$-uniform tumbleweed with the SLP. Therefore, by our inductive hypothesis, $\mathcal{H}_1(x)$ is an $(n-1)$-uniform economical extremal hypergraph, so Maker has an $(n-1)$-turn winning strategy on $\mathcal{H}_1(x) = \mathcal{H}_1(x, \overline{y_1})$. Thus, our note above implies that Maker has an n-turn winning strategy on $\mathcal{H}$.

Therefore, a hypergraph $\mathcal{H}$ is an n-uniform economical extremal hypergraph if and only if $\mathcal{H}$ is an n-uniform tumbleweed with the Single Label Property. □

Corollary 19 *Corollary to Lemma 13: If $\mathcal{H}$ is an n-uniform tumbleweed and x and x' are a heavy pair in $\mathcal{H}$, then there exists a good labeling L of a complete n-level binary tree T which realizes $\mathcal{H}$ in which x and x' appear as the labels of the nodes in level 2 of T.*

Proof of Corollary 19: We essentially repeat the beginning of the proof of Theorem 18. Let $\mathcal{H}$ be an n-uniform tumbleweed and let x and x' be a heavy pair in $\mathcal{H}$. Let $\mathcal{H}_1^{(1)} = \mathcal{H}_1(x)$

and $\mathcal{H}_1^{(2)} = \mathcal{H}_1(x')$. By Lemma 13, we know that $\mathcal{H}_1^{(i)}$ is an $(n-1)$-uniform tumbleweed for $i \in \{1,2\}$. Let $L^{(i)}$ be a good labeling on an $(n-1)$-level complete binary tree $T^{(i)}$ which realizes $\mathcal{H}_1^{(i)}$ for each $i \in \{1,2\}$. Let T be the n-level complete binary tree whose root node is N_r and whose left and right subtrees are $T^{(1)}$ and $T^{(2)}$, respectively. Let $L : V(T) \to \mathbb{N}$ be the labeling of T in which $L(N_r) = r$ and $L(N) = L^{(i)}(N)$ for all $N \in V(T^{(i)})$ for $i \in \{1,2\}$. Since x is the root of $\mathcal{H}_1(x)$ and x' is the root of $\mathcal{H}_1(x')$, then Lemma 7 implies that the level 2 nodes of T are labeled with x and x'. By repeating exactly the same argument from the proof of Theorem 18, we can conclude that $A \in \mathcal{H}$ if and only if there is a root-to-leaf path in T which realizes A when T is labeled with L. Thus, L is a labeling of T which realizes $\mathcal{H}$ in which x and x' appear as the labels of the nodes in level 2 of T, and by Theorem 14, L must be a good labeling. $\square$

Lemma 20 *If $\mathcal{H}$ is an n-uniform tumbleweed with the Single Label Property, then $|\mathcal{H}(v)| = 2^{n-k}$ for each vertex $v \in V(\mathcal{H})$, where $k = \left|\cap_{A \in \mathcal{H}(v)} A\right|$.*

Proof of Lemma 20: Let $\mathcal{H}$ be an n-uniform tumbleweed with the SLP. Let $v \in V(\mathcal{H})$ be an arbitrary vertex. Then there is a good labeling L of a complete n-level binary tree T which realizes $\mathcal{H}$ in which v appears exactly once. Let N be the unique node in T which is labeled v. Then the set of edges $\mathcal{H}(v)$ corresponds precisely to the root-to-leaf paths in T which pass through N. Suppose N is in level k, then there are exactly 2^{n-k} root-to-leaf paths which pass through N, thus, $|\mathcal{H}(v)| = 2^{n-k}$.

Let T_N be the subtree rooted at N. Lemma 3 implies that L restricted to T_N realizes a tumbleweed $\mathcal{H}_N$. Let $v_1, \ldots, v_{k-1}$ be the labels on the path from the root of T to the parent of N. Notice that each $A \in \mathcal{H}(v)$ satisfies $A = \{v_1, \ldots, v_{k-1}\} \cup A_N$ where $A_N \in \mathcal{H}_N$. Moreover, since v is the root of $\mathcal{H}_N$, we have $\left|\cap_{A \in \mathcal{H}(v)} A\right| = k$. $\square$

Since Lemma 20 gives a necessary condition for a tumbleweed $\mathcal{H}$ to have the SLP, we will say that an n-uniform tumbleweed is *amenable to the SLP* if for each vertex $v \in V(\mathcal{H})$, $|\mathcal{H}(v)| = 2^{n-k}$ where $k = \left|\cap_{A \in \mathcal{H}(v)} A\right|$. It turns out that the converse of Lemma 20 also holds. Thus, once we prove the converse, we will have an alternative characterization of the economical extremal hypergraphs for the Erdős–Selfridge theorem. We introduce some notation and a lemma which aid in our proof of the converse. For a hypergraph $\mathcal{H}$ and a set $\{v_1, \ldots, v_k\}$, we let $\mathcal{H}(v_1, \ldots, v_k) = \{A \in \mathcal{H} : \{v_1, \ldots, v_k\} \subseteq A\}$ be the set of edges in $\mathcal{H}$ which contain $\{v_1, \ldots, v_k\}$.

Lemma 21 *Let $\mathcal{H}$ be an n-uniform tumbleweed. If $\{v_1, \ldots, v_k\}$ is a k-subset of $V(\mathcal{H})$, then $|\mathcal{H}(v_1, \ldots, v_k)| \leq 2^{n-k}$.*

Proof of Lemma 21: Let $\mathcal{H}$ be an n-uniform tumbleweed. We proceed by induction on n. When $n = 1$, the result is trivial.

Let $n \geq 2$, and assume the result for ℓ-uniform tumbleweeds where $\ell < n$. Let $\{v_1, \ldots, v_k\}$ be a k-subset of $V(\mathcal{H})$ which satisfies $|\mathcal{H}(v_1, \ldots, v_k)| \geq |\mathcal{H}(w_1, \ldots, w_k)|$ for all k-subsets $\{w_1, \ldots, w_k\} \subseteq V(\mathcal{H})$. We will show that $|\mathcal{H}(v_1, \ldots, v_k)| \leq 2^{n-k}$, thus proving the result for all k-subsets. We may assume that the root r of $\mathcal{H}$ is an element of $\{v_1, \ldots, v_k\}$, since for an arbitrary k-subset $\{w_1, w_2, \ldots, w_k\}$ we have $\mathcal{H}(r, w_2, \ldots, w_k) = \mathcal{H}(w_2, \ldots, w_k) \supseteq \mathcal{H}(w_1, w_2, \ldots, w_k)$. Without loss of generality, we may assume that $v_1 = r$.

Let x and x' be a heavy pair in $\mathcal{H}$. Since $\mathcal{H}(x)$ and $\mathcal{H}(x')$ partition the edges of $\mathcal{H}$, we can write

$$\begin{aligned}\mathcal{H}(r, v_2, \ldots, v_k) &= \mathcal{H}(v_2, \ldots, v_k)\\ &= \mathcal{H}(x, v_2, \ldots, v_k) \cup \mathcal{H}(x', v_2, \ldots, v_k)\end{aligned}$$

where the union is disjoint. Thus,

$$|\mathcal{H}(r, v_2, \ldots, v_k)| = |\mathcal{H}(x, v_2, \ldots, v_k)| + |\mathcal{H}(x', v_2, \ldots, v_k)|. \tag{3}$$

Recall that $\mathcal{H}_1 = \{A - r : A \in \mathcal{H}\}$, and let $\mathcal{H}_1^{(1)} = \mathcal{H}_1(x)$ and $\mathcal{H}_1^{(2)} = \mathcal{H}_1(x')$. Note that $|\mathcal{H}(x, v_2, \ldots, v_k)| = |\mathcal{H}_1^{(1)}(x, v_2, \ldots, v_k)|$ because we simply remove r from every edge, and while the x is almost redundant in the right-hand-side of the equality, the equality is true nonetheless. Similarly, we note that $|\mathcal{H}(x', v_2, \ldots, v_k)| = |\mathcal{H}_1^{(2)}(x', v_2, \ldots, v_k)|$. By Lemma 13, $\mathcal{H}_1^{(i)}$ is an $(n-1)$-uniform tumbleweed for $i \in \{1, 2\}$.

If $v_i \notin \{x, x'\}$ for $2 \le i \le k$, then $\{x, v_2, \ldots, v_k\}$ and $\{x', v_2, \ldots, v_k\}$ are both k-sets, and by the inductive hypothesis, $|\mathcal{H}_1^{(1)}(x, v_2, \ldots, v_k)| \le 2^{(n-1)-k}$ and $|\mathcal{H}_1^{(2)}(x', v_2, \ldots, v_k)| \le 2^{(n-1)-k}$. When we combine these bounds with our two notes above and equation (3), we conclude that $|\mathcal{H}(r, v_2, \ldots, v_k)| \le 2^{n-k}$ as desired.

If $v_i = x$ for some $2 \le i \le k$, then $\{x, v_2, \ldots, v_k\}$ is a $(k-1)$-set and by the inductive hypothesis, $|\mathcal{H}_1^{(1)}(x, v_2, \ldots, v_k)| \le 2^{(n-1)-(k-1)} = 2^{n-k}$. Yet, when $v_i = x$, Lemma 4 implies that $\mathcal{H}(x', v_2, \ldots, v_k) = \emptyset$, thus, $|\mathcal{H}(x', v_2, \ldots, v_k)| = 0$. Therefore, by equation (3) and our two notes, we again conclude that $|\mathcal{H}(r, v_2, \ldots, v_k)| \le 2^{n-k}$. A similar argument can be made for the case when $v_i = x'$ for some $2 \le i \le k$. □

Corollary 22 *Let $\mathcal{H}$ be an n-uniform tumbleweed. If $\left|\bigcap_{A \in \mathcal{H}(v)} A\right| = k$, then $|\mathcal{H}(v)| \le 2^{n-k}$.*

Proof of Corollary 22: Let $\mathcal{H}$ be an n-uniform tumbleweed. Let $\{v_1, \ldots, v_k\} = \cap_{A \in \mathcal{H}(v)} A$, then $\mathcal{H}(v_1, \ldots, v_k) = \mathcal{H}(v)$. If $|\{v_1, \ldots, v_k\}| = k$, then Lemma 21 implies $|\mathcal{H}(v_1, \ldots, v_k)| \le 2^{n-k}$, thus, $|\mathcal{H}(v)| \le 2^{n-k}$. □

Lemma 23 *If $\mathcal{H}$ is an n-uniform tumbleweed such that $|\mathcal{H}(v)| = 2^{n-k}$ for each vertex $v \in V(\mathcal{H})$, where $k = \left|\cap_{A \in \mathcal{H}(v)} A\right|$, then $\mathcal{H}$ satisfies the Single Label Property.*

Proof of Lemma 23: Let $\mathcal{H}$ be an n-uniform tumbleweed such that each vertex $v \in V(\mathcal{H})$ satisfies $|\mathcal{H}(v)| = 2^{n-k}$ where $k = \left|\cap_{A \in \mathcal{H}(v)} A\right|$. We will show that if v is an arbitrary vertex such that $|\mathcal{H}(v)| = 2^{n-k}$ where $k = \left|\cap_{A \in \mathcal{H}(v)} A\right|$, then there is a good labeling of a complete n-level binary tree T which realizes $\mathcal{H}$ in which v is used as a label exactly once. We proceed by induction on n and k.

When $n = 1$ and $n = 2$, it is easy to check that the result holds for all k. When n is arbitrary and $k = 1$, we have $|\mathcal{H}(v)| = 2^{n-1}$, thus $v = r$, the root of $\mathcal{H}$. By Lemma 7, exactly one node of T is labeled r in every labeling which realizes $\mathcal{H}$. When $n \ge 2$ is arbitrary and $k = 2$, we have $|\mathcal{H}(v)| = 2^{n-2}$ and $|\cap_{A \in \mathcal{H}(v)} A| = 2$. This implies that v is not the root of $\mathcal{H}$, and Lemma 8 implies that v and its sibling v' are a heavy pair in $\mathcal{H}$. Corollary 19 implies that there is a good labeling of T in which v and v' appear as the labels of the level 2 nodes of T, in which case, v appears as a label exactly once.

Let n and k be arbitrary such that $n \geq k \geq 3$. Our inductive hypotheses are: if $\mathcal{F}$ is an m-uniform tumbleweed that is amenable to the SLP, where $m < n$, then $\mathcal{F}$ satisfies the SLP, and if $\mathcal{F}$ is an n-uniform tumbleweed that is amenable to the SLP, then for each $v \in V(\mathcal{F})$ such that $|\cap_{A \in \mathcal{F}(v)} A| = \ell < k$, there is a good labeling of T which realizes $\mathcal{F}$ in which v appears as a label exactly once.

Let $\mathcal{H}$ be an n-uniform tumbleweed that is amenable to the SLP, and let v satisfy $|\cap_{A \in \mathcal{H}(v)} A| = k$. Let $\{v_1, v_2, \ldots, v_k\} = \cap_{A \in \mathcal{H}(v)} A$, where $|\mathcal{H}(v_1)| \geq |\mathcal{H}(v_2)| \geq \cdots \geq |\mathcal{H}(v_k)|$. Since $\mathcal{H}(v) \subseteq \mathcal{H}(v_i)$, then $|\mathcal{H}(v)| \leq |\mathcal{H}(v_i)|$ for $1 \leq i \leq k$. Moreover, $v \in \{v_1, \ldots, v_k\}$, so let us assume (without loss of generality) that $v_k = v$. Thus, $\mathcal{H}(v_k) \subseteq \mathcal{H}(v_i)$ for $1 \leq i \leq k$. Notice that we cannot have $\mathcal{H}(v_i) = \mathcal{H}(v_j)$ for $i \neq j$ because Lemma 6 would imply that $v_i = v_j$, yet the v_i's must be distinct.

Claim 24 *(If we assume the inductive hypotheses, then)* $|\mathcal{H}(v_i)| \geq 2^{n-i}$ *for all* $1 \leq i \leq k$.

Proof of Claim 24: Assume towards a contradiction that the claim is not true, and let j be such that $|\mathcal{H}(v_i)| \geq 2^{n-i}$ for all $1 \leq i \leq j-1$, but $|\mathcal{H}(v_j)| = 2^{n-\ell} < 2^{n-j}$. We cannot have $j = k$, or else we contradict the fact that $|\mathcal{H}(v_k)| = 2^{n-k}$. If $j = 1$, then $|\mathcal{H}(v_i)| < 2^{n-1}$ for $1 \leq i \leq k$, which contradicts the fact that $\{v_1, \ldots, v_k\}$ must contain the root of $\mathcal{H}$. Additionally, we cannot have $\ell = k$, or else we have $|\mathcal{H}(v_k)| = |\mathcal{H}(v_j)|$ and $\mathcal{H}(v_k) \subseteq \mathcal{H}(v_j)$, which would imply $\mathcal{H}(v_k) = \mathcal{H}(v_j)$, yet $v_k \neq v_j$. Thus, $1 < j < \ell < k$.

Since $\ell < k$, our inductive hypothesis implies that there is a good labeling L_j of T, in which v_j appears as a label exactly once. Since $|\mathcal{H}(v_j)| = 2^{n-\ell}$, then v_j is the label of a node in level ℓ of T. Let $(w_1, \ldots, w_{\ell-1}, v_j)$ be the ordered labels of the path from the root of T to the unique node labeled v_j. Thus, w_i is the label of a node in level i of T, which implies that $|\mathcal{H}(w_i)| \geq 2^{n-i}$, for $1 \leq i \leq \ell - 1$. Since L_j realizes $\mathcal{H}$, we conclude that $\{w_1, \ldots, w_{\ell-1}, v_j\} \subseteq A$ for all $A \in \mathcal{H}(v_j)$. Since $\mathcal{H}(v_k) \subseteq \mathcal{H}(v_j)$, we also conclude that $\{w_1, \ldots, w_{\ell-1}\} \subseteq \{v_1, \ldots, v_k\}$. However, we know that $|\mathcal{H}(v_i)| \leq 2^{n-\ell}$ for $j \leq i \leq k$, and $|\mathcal{H}(w_i)| \geq 2^{n-\ell+1}$ for $1 \leq i \leq \ell - 1$. Thus, we must, in fact, have $\{w_1, \ldots, w_{\ell-1}\} \subseteq \{v_1, \ldots, v_{j-1}\}$, which is impossible since $j < \ell$. $\square$

Claim 24 applied to v_1 implies that $v_1 = r$, the root of $\mathcal{H}$. Since $v_2 \neq v_1$, Claim 24 and Lemma 8 imply that v_2 and its sibling v_2' are a heavy pair in $\mathcal{H}$. Corollary 19 implies that there is a good labeling L of T in which v_2 and v_2' appear as the labels of the level 2 nodes N and N' of T, respectively. Since $\mathcal{H}(v_2)$ and $\mathcal{H}(v_2')$ partition $\mathcal{H}$, and $\mathcal{H}(v) \subseteq \mathcal{H}(v_2)$, only nodes in T_N, the subtree rooted at N, are labeled with v. Let $\mathcal{H}' = \mathcal{H}_1(v_2)$ and $\mathcal{H}'' = \mathcal{H}_1(v_2')$. By Lemma 13, $\mathcal{H}'$ and $\mathcal{H}''$ are $(n-1)$-uniform tumbleweeds. Notice that L restricted to T_N realizes $\mathcal{H}'$. We will show that $\mathcal{H}'$ is amenable to the SLP, thus, by induction, $\mathcal{H}'$ satisfies the SLP. This will allow us to relabel T_N so that v is used as a label exactly once in T_N, hence, exactly once in T.

Let $w \in V(\mathcal{H}')$ be arbitrary. Suppose $|\cap_{A \in \mathcal{H}(w)} A| = \ell$ and $\cap_{A \in \mathcal{H}(w)} A = \{w_1, \ldots, w_\ell\}$, where $w_1 = r$, the root of $\mathcal{H}$. Thus, $\cap_{A \in \mathcal{H}'(w)} A = \{v_2, w_2, \ldots, w_\ell\}$, since every edge in $\mathcal{H}'(w)$ has r removed, but must contain v_2, the root of $\mathcal{H}'$. (It may be the case that $v_2 = w_i$ for some $2 \leq i \leq \ell$.) Since $\mathcal{H}$ is amenable to the SLP, then $|\mathcal{H}(w)| = 2^{n-\ell}$. Let $n' = n - 1$ and let $\ell' = |\{v_2, w_2, \ldots, w_\ell\}|$. We must show that $|\mathcal{H}'(w)| = 2^{n'-\ell'}$. If $v_2 = w_i$ for some $2 \leq i \leq \ell$, then $\ell' = \ell - 1$, and $|\mathcal{H}'(w)| = |\mathcal{H}(w)| = 2^{n-\ell} = 2^{n'-\ell'}$.

Now suppose that $v_2 \neq w_i$, for any $2 \leq i \leq \ell$, so that $\ell' = \ell$. Notice that $\mathcal{H}(w) = \mathcal{H}(w, v_2) \cup \mathcal{H}(w, v_2')$, where the union is disjoint. If $|\mathcal{H}(w, v_2)| = |\mathcal{H}(w, v_2')|$, then $|\mathcal{H}(w, v_2)| = 2^{n-\ell-1}$. Since $|\mathcal{H}'(w)| = |\mathcal{H}(w, v_2)|$, this implies $|\mathcal{H}'(w)| = 2^{n'-\ell'}$. If $|\mathcal{H}(w, v_2)| \neq |\mathcal{H}(w, v_2')|$, then we reach a contradiction. First, we consider the case where $|\mathcal{H}(w, v_2)| > 2^{n-\ell-1}$. Again, since $|\mathcal{H}'(w)| = |\mathcal{H}(w, v_2)|$, this implies $|\mathcal{H}'(w)| > 2^{n'-\ell'}$. However, this is a contradiction to Corollary 22 applied to $\mathcal{H}'$ and $\mathcal{H}'(w)$. When $|\mathcal{H}(w, v_2')| > 2^{n-\ell-1}$, we can apply the same arguments to $\mathcal{H}(w, v_2')$ and $\mathcal{H}''(w)$. (We know that $v_2' \notin \{w_2, \dots, w_\ell\}$ because if $v_2' \in \{w_2, \dots, w_\ell\}$, then since $\mathcal{H}'(w) = \mathcal{H}'(v_2, w_2, \dots, w_\ell)$, every edge $A \in \mathcal{H}'(w)$ contains v_2 and v_2', which contradicts Lemma 4 applied to v_2, v_2' and $\mathcal{H}$ when we extend $A \in \mathcal{H}'(w)$ to $A \cup \{r\} \in \mathcal{H}$.) Therefore, $\mathcal{H}'$ must be amenable to the SLP.

Since $\mathcal{H}'$ is amenable to the SLP, then, by induction, $\mathcal{H}'$ satisfies the SLP. Let L' be a good labeling on T_N, in which v appears as a label exactly once, which realizes $\mathcal{H}' = \mathcal{H}_1(v_2)$. We create a labeling L_v on T, in which v appears as a label exactly once, which realizes $\mathcal{H}$ by letting $L_v(N) = L'(N)$ if $N \in V(T_N)$ and $L_v(N) = L(N)$ if $N \notin V(T_N)$. By Theorem 14, L_v is a good labeling. $\square$

Corollary 25 *An n-uniform hypergraph $\mathcal{H}$ is an economical extremal hypergraph for the Erdős–Selfridge theorem if and only if $\mathcal{H}$ is a tumbleweed such that for all $v \in V(\mathcal{H})$, $|\mathcal{H}(v)| = 2^{n-k}$ where $k = \left|\bigcap_{A \in \mathcal{H}(v)} A\right|$.*

Proof of Corollary 25: This follows from Lemmas 20 and 23, and Theorem 18.

5 Conclusion

Theorem 18 and Corollary 25 provide two distinct characterizations of the economical extremal hypergraphs for the Erdős–Selfridge theorem. In [7], Sundberg proved that there is a unique extremal hypergraph for the $(p : q)$-Erdős–Selfridge theorem when $q \geq 2$. However, the problem of characterizing all extremal hypergraphs for the Erdős–Selfridge theorem is still a wide open problem. Perhaps the next avenue to pursue is characterizing which labelings of a complete binary tree realize an extremal hypergraph for the Erdős–Selfridge theorem.

As a closing remark, we note that the results in this paper reveal two straightforward algorithms: one which, given a tumbleweed $\mathcal{H}$ (say, as a set of edges), will produce a good labeling of a complete binary tree which realizes $\mathcal{H}$; and another which, given an economical extremal hypergraph for the Erdős–Selfridge theorem $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$, will produce a good labeling of a complete binary tree which realizes $\mathcal{H}$ in which v appears exactly once. We leave it to the reader to provide the details.

6 Acknowledgements

The authors would like to thank the Undergraduate Research Center at Occidental College for partially funding Emily Heath's participation in this research. We also wish to thank József Beck and Klay Kruczek for their valuable input and suggestions, and Greg Tollisen, Tamás Lengyel, and the anonymous referees for making many helpful suggestions and comments that improved the presentation of this paper.

References

[1] J. Beck. Remarks on positional games. I. *Acta Math. Acad. Sci. Hungar.*, 40(1-2):65-71, 1982.

[2] József Beck. Combinatorial Games: Tic-Tac-Toe Theory, Cambridge University Press, 2008.

[3] József Beck. Inevitable randomness in discrete mathematics. *University Lecture Series, 49* American Mathematical Society, Providence, RI, xii+250, 2009.

[4] P. Erdős and J. L. Selfridge. On a combinatorial game. *J. Combinatorial Theory Ser. A*, 14:298-301, 1973.

[5] Xiaoyun Lu. A characterization on n-critical economical generalized tic-tac-toe games. *Discrete Math.*, 110(1-3):197-203, 1992.

[6] E. L. Sundberg. Economical tight examples for the biased Erdős-Selfridge theorem. *Discrete Math.*, 308 (15): 3308-3314, 2008.

[7] E. L. Sundberg. Extremal hypergraphs for the biased Erdős–Selfridge theorem. *The Electronic Journal of Combinatorics.*, 20 (1): # P26, 2013.