Bin-based pairing strategies for the Maker–Breaker game on the Boolean hypercube with subcubes as winning sets.

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Abstract

We consider the Maker–Breaker positional game on the vertices of the n-dimensional hypercube $\{0,1\}^n$ with k-dimensional subcubes as winning sets. We describe a pairing strategy which allows Breaker to win when $k = n/4 + 1$ in the case where n is a power of 4. Our results also imply the general result that there is a Breaker's win pairing strategy for any $n \geq 3$ if $k = \left\lfloor \frac{3}{7}n \right\rfloor + 1$.

1 Introduction

A positional game is a generalization of Tic-Tac-Toe played on a hypergraph $(V, \mathcal{H})$ where the vertices can be considered the "board" on which the game is played, and the edges can be thought of as the "winning sets." A *positional game* on $(V, \mathcal{H})$ is a two-player game where at every turn each player alternately occupies a previously unoccupied vertex from V. In a *strong positional game*, the first player to occupy all vertices of some edge $A \in \mathcal{H}$ wins. If at the end of play no edge is completely occupied by either player, that play is declared a draw. Normal 3×3 Tic-Tac-Toe is a strong positional game where the vertices of the hypergraph are the nine positions and the edges are the eight winning lines. In a *Maker–Breaker* positional game, the first player, *Maker*, wins if she[1] occupies all vertices of some edge $A \in \mathcal{H}$, otherwise the second player, *Breaker*, wins. Therefore, by definition there are no draw plays in Maker–Breaker games. We say that a player P has a *winning strategy* if no matter how the other player plays, player P wins by following that winning strategy. It is well-known that in a finite Maker–Breaker game, exactly one player has a winning strategy. (For a nice introduction to positional games, please see [5], [6], and [22].)

[1] In this paper, we will refer to Maker with feminine pronouns, such as "she" and "her," and we will refer to Breaker with masculine pronouns, such as, "he."

Recall that the n-dimensional boolean hypercube Q_n is a bipartite graph whose vertex set is $\{0,1\}^n$ and whose edge set is the set of all pairs of vertices that differ in exactly one coordinate. A k-dimensional subcube of Q_n is formed by selecting $n-k$ coordinates to be fixed, choosing fixed values for each of those coordinates, then allowing the remaining k coordinates to take on all 2^k possible values. Thus, each k-dimensional subcube of Q_n has cardinality 2^k, and there are $\binom{n}{n-k}2^{n-k}$ distinct k-dimensional subcubes contained in Q_n. Let $\mathcal{Q}(n,k)$ denote the hypergraph whose vertex set is $\{0,1\}^n$ and whose edge set is the set of all k-dimensional subcubes of Q_n. In [23], Sundberg and Kruczek introduce a Maker–Breaker game played on $\mathcal{Q}(n,k)$, and describe pairing strategies for Breaker, i.e., strategies where, using a set M of pairwise disjoint pairs of vertices, Breaker moves as follows: each time Maker occupies a vertex x, if there is a pair in M which contains x, then Breaker immediately responds by occupying the other vertex in that pair (if the other vertex is currently unoccupied), otherwise, Breaker occupies an arbitrary (unoccupied) vertex. This guarantees that Breaker will occupy at least one vertex from each pair in M. If every winning set (in our case k-dimensional subcube) contains at least one pair from M, then Breaker will win by using M as a pairing strategy. The sets of pairs that we use in our Breaker's win pairing strategies will always be sets of edges. This is a greedy approach because the number of subcubes which contain a pair of vertices is maximized if those two vertices differ in exactly one coordinate. Since the sets of pairs must be disjoint, *our pairing strategies correspond to matchings* in Q_n.

Let $ps(n)$ be the smallest value of k such that Breaker wins the positional game on $\mathcal{Q}(n,k)$ by *using a pairing strategy*. Proposition 9 in [14], implies that $ps(n) > \ln(n)$. Sundberg and Kruczek showed that $ps(n) \le n-3$. We improve their result by describing a Breaker's win pairing strategy which results in an upper bound on $ps(n)$ of $n/4+1$ when n is a power of 4.

The remainder of the paper is organized as follows. In Section 2, we explain the main techniques behind constructing our Breaker's win pairing strategies through an illustrative example. In Section 3, we state and prove a theorem which uses those techniques and can be used to produce an upper bound on $ps(n)$ of $n/3+1$ if $n = 6\cdot 4^d$ or $n = 9\cdot 4^d$ for some $d \ge 1$. In Section 4, we enhance the techniques from Section 3 to prove Theorem 3, which yields an upper bound on $ps(n)$ of $n/4+1$ when n is a power of 4. In Section 5, we briefly discuss Breaker's win pairing strategies for specific values of n and k, including the result that $ps(n) \le \frac{3}{7}n+1$ for all $n \ge 3$. In Section 6, we prove a generalization of a lemma from Section 4 in order to allow us to prove Theorem 4, which is similar to Theorem 3, yet yields an upper bound on $ps(n)$ of $n/3+1$ when n is a power of 3.

In Section 7, we briefly mention how our results from Theorems 3 and 4 can be viewed as a variation of d-polychromatic edge colorings of Q_n.

2 Illustrative Example

We will generalize a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$ to create Breaker's win pairing strategies for games with board dimension at least 12. The basic strategy will be defined on boards with dimension $4n$. We will partition the $4n$ coordinates into four bins of size n, and make use of a Breaker's win pairing strategy for $\mathcal{Q}(n,k)$. Since the first useful (non-trivial)

Breaker's win pairing strategy is for $\mathcal{Q}(3, 2)$, the smallest bins we use are of size 3. Thus, we will use our generalized strategy on games with board dimension at least 12.

We will use the following Breaker's win pairing strategy for $\mathcal{Q}(4, 2)$:

$$\begin{aligned} PS(4,2) = \{&(v,0,0,0),(0,v,1,0),(0,0,v,1),(0,1,0,v),\\ &(v,1,1,1),(1,v,0,1),(1,1,v,0),(1,0,1,v)\}, \end{aligned}$$

where, for example, $(v, 0, 0, 0)$ indicates the edge with endpoints $(0, 0, 0, 0)$ and $(1, 0, 0, 0)$.

To denote a d-dimensional subcube of Q_n, we will write a vector with n coordinates, where d of the coordinates are $*$'s and each of the other $n - d$ coordinates is fixed with a value of 0 or 1. For example, $(*, 0, 1, *)$ is the 2-dimensional subcube of Q_4 with the following set of vertices $\{(0, 0, 1, 0), (0, 0, 1, 1), (1, 0, 1, 0), (1, 0, 1, 1)\}$. We will say that a pair or edge *handles* a subcube if the subcube contains that edge. For example, $(1, 0, 1, v)$ handles $(*, 0, 1, *)$. Moreover, $(*, 0, 1, *), (1, *, 1, *)$, and $(1, 0, *, *)$ are precisely the 2-dimensional subcubes which are handled by $(1, 0, 1, v)$. To obtain these 2-dimensional subcubes, we simply replace v with a "$*$," then choose one of the three remaining coordinates to also become a "$*$." We will say that a set of edges *handles* a subcube S if it contains an edge that handles S.

While $PS(4, 2)$ has eight edges, each pairing strategy we describe will be the union of eight sets of edges, where each set is based on one of the edges from $PS(4, 2)$. As an example, let us describe our Breaker's win pairing strategy for $\mathcal{Q}(12, 5)$ which is derived from $PS(4, 2)$. We will partition the 12 coordinates of each vector we use into four bins where the first bin contains coordinates 1-3, the second bin contains coordinates 4-6, the third bin contains coordinates 7-9, and the fourth bin contains coordinates 10-12. We will use the following sets to define each of the eight sets of edges in our Breaker's win pairing strategy. Let $\mathbf{0}_3 = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$, i.e., the set of even parity vectors from $\{0, 1\}^3$. Let $\mathbf{1}_3 = \{(1, 1, 1), (0, 0, 1), (0, 1, 0), (1, 0, 0)\}$, i.e., the set of odd parity vectors from $\{0, 1\}^3$. Let $\mathbf{v}_3 = \{(v, 0, 1), (1, v, 0), (0, 1, v)\}$, i.e., a Breaker's win pairing strategy for $\mathcal{Q}(3, 2)$. (When the dimensions of the vectors in these sets are known, we will frequently drop the subscript.) Each 12-dimensional vector in our Breaker's win pairing strategy will be formed by "gluing together" or concatenating four vectors from the sets $\mathbf{0}, \mathbf{1}, \mathbf{v}$. We will determine which vectors to glue together by using $PS(4, 2)$ as a guide. For example, our Breaker's win pairing strategy will contain a set of edges that corresponds to edge $(0, 1, 0, v) \in PS(4, 2)$, specifically,

$$(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v}) = \{(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) : \vec{x}_1 \in \mathbf{0}, \vec{x}_2 \in \mathbf{1}, \vec{x}_3 \in \mathbf{0}, \vec{x}_4 \in \mathbf{v}\}, \tag{1}$$

where we really mean the set of 12-dimensional vectors which correspond to the set described in equation (1). For example, because $(0, 0, 0) \in \mathbf{0}$, $(0, 1, 0) \in \mathbf{1}, (1, 1, 0) \in \mathbf{0}$, and $(1, v, 0) \in \mathbf{v}$, then $(0, 0, 0, 0, 1, 0, 1, 1, 0, 1, v, 0) \in (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{v})$, even though equation (1) technically describes vectors of the form $((0, 0, 0), (0, 1, 0), (1, 1, 0), (1, v, 0))$. Our Breaker's win pairing strategy for $\mathcal{Q}(12, 5)$ will be formed by taking the union of the sets in the following set:

$$\begin{aligned} \mathit{BinSetsPS}(4,2) = \{&(\mathbf{v},\mathbf{0},\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{v},\mathbf{1},\mathbf{0}),(\mathbf{0},\mathbf{0},\mathbf{v},\mathbf{1}),(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v}),\\ &(\mathbf{v},\mathbf{1},\mathbf{1},\mathbf{1}),(\mathbf{1},\mathbf{v},\mathbf{0},\mathbf{1}),(\mathbf{1},\mathbf{1},\mathbf{v},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{v})\}, \end{aligned}$$

i.e., our Breaker's win pairing strategy is

$$\begin{aligned} BinPS(4,2) = &(\mathbf{v},\mathbf{0},\mathbf{0},\mathbf{0}) \cup (\mathbf{0},\mathbf{v},\mathbf{1},\mathbf{0}) \cup (\mathbf{0},\mathbf{0},\mathbf{v},\mathbf{1}) \cup (\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v}) \\ &\cup (\mathbf{v},\mathbf{1},\mathbf{1},\mathbf{1}) \cup (\mathbf{1},\mathbf{v},\mathbf{0},\mathbf{1}) \cup (\mathbf{1},\mathbf{1},\mathbf{v},\mathbf{0}) \cup (\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{v}). \end{aligned}$$

Let us refer to the sets $\mathbf{0}, \mathbf{1}, \mathbf{v}$ as *bin-sets*, and refer to the sets in $BinSetsPS(4,2)$ as *bin-forms*, and say, for example, that bin-form $(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v})$ has bin-sets $\mathbf{0},\mathbf{1},\mathbf{0}$, and $\mathbf{v}$. Observe that the cardinality of each of the eight bin-forms in our Breaker's win pairing strategy is $3 \cdot 4^3 = 192$, because each of these bin-forms contains three bin-sets of cardinality 4 and one bin-set of cardinality 3. Thus, our Breaker's win pairing strategy contains exactly $8(192) = 1536$ edges.

Let us say that a vertex or an edge is *consistent* with a subcube if it is an element of that subcube. One way to check if a vector (vertex or edge) is consistent with a subcube is to do the following: for each coordinate of the subcube that contains a "$*$," we place a "$*$" in the corresponding coordinate of the vector. If after making these substitutions we are left with the vector representation of the subcube, then the vector (vertex or edge) is consistent with the subcube. For example, $(0,0,0,0,1,0,1,1,0,1,v,0)$ is consistent with $(0,0,0,*,1,*,1,*,0,*,*,0)$. A key observation for us will be that $(0,0,0,0,1,0,1,1,0,1,v,0)$ is consistent with $(0,0,0,*,1,*,1,*,0,*,*,0)$ if and only if $((0,0,0),(0,1,0),(1,1,0),(1,v,0))$ is "consistent" with $((0,0,0),(*,1,*),(1,*,0),(*,*,0))$, i.e., if and only if $(0,0,0)$ is consistent with $(0,0,0)$, and $(0,1,0)$ is consistent with $(*,1,*)$, and $(1,1,0)$ is consistent with $(1,*,0)$, and $(1,v,0)$ is consistent with $(*,*,0)$.

To determine which subcubes our Breaker's win pairing strategy can handle, we will focus on how many coordinates are fixed by the subcube in each bin. We also make the following important observations. Since Q_3 is a bipartite graph with partite sets $\mathbf{0}$ and $\mathbf{1}$, each of $\mathbf{0}$ and $\mathbf{1}$ is a vertex cover for the edges of Q_3. In our language, for any subcube $S \subseteq Q_3$ with dimension at least 1, i.e., 0, 1, or 2 fixed coordinates, there is a vertex in $\mathbf{0}$ (respectively, in $\mathbf{1}$) which is consistent with S. Since $\mathbf{v}$ is a Breaker's win pairing strategy for $\mathcal{Q}(3,2)$, for any subcube $S \subseteq Q_3$ with dimension at least 2, i.e., 0 or 1 fixed coordinates, there is an edge in $\mathbf{v}$ which is consistent with S.

Using the same 5-dimensional subcube $(0,0,0,*,1,*,1,*,0,*,*,0)$ as an example, let us describe how we can find an edge in our Breaker's win pairing strategy that is consistent with a given subcube. We begin by partitioning $(0,0,0,*,1,*,1,*,0,*,*,0)$ into $(0,0,0),(*,1,*),(1,*,0)$, and $(*,*,0)$ according to the four bins we described above. We want to show that at least one of the eight bin-forms from $BinSetsPS(4,2)$ contains an edge that is consistent with our subcube. As we noted above, such an edge will have to be consistent with our subcube in each bin. For the first bin, we need to use a bin-set which contains a vector that is consistent with $(0,0,0)$. Since there is only one such bin-set, we need $\mathbf{0}$ to be the bin-set in the first bin. For the second bin, we need to use a bin-set which contains a vector that is consistent with $(*,1,*)$. Since only one coordinate is fixed in $(*,1,*)$ we can use any bin-set ($\mathbf{0},\mathbf{1}$, or $\mathbf{v}$) in the second bin. Since $(1,*,0)$ has two fixed coordinates, we are guaranteed that $\mathbf{0}$ and $\mathbf{1}$ each contain a vertex that is consistent with $(1,*,0)$, thus, we can use either $\mathbf{0}$ or $\mathbf{1}$ in the third bin. (By coincidence, $(1,v,0) \in \mathbf{v}$ is consistent with $(1,*,0)$, but we cannot rely on coincidence for our proof, so we will not permit ourselves to use $\mathbf{v}$ for the third bin-set.) Since $(*,*,0)$ has only one fixed coordinate, we can use any bin-set for the fourth bin. Thus, we should be able to find a vector which is consistent with our subcube in either bin-form $(\mathbf{0}, \mathbf{v}, \mathbf{1}, \mathbf{0})$ or bin-form $(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{v})$. Indeed, we were already aware of $(0, 0, 0, 0, 1, 0, 1, 1, 0, 1, v, 0) \in (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{v})$ which is consistent with our subcube. Let us show how to find an edge in $(\mathbf{0}, \mathbf{v}, \mathbf{1}, \mathbf{0})$ which is consistent with our subcube. We need $(0, 0, 0)$ in the first bin of our edge. For the second bin, we need an element of $\mathbf{v}$ which is consistent with $(*, 1, *)$, thus, we use $(0, 1, v)$ in the second bin. For the third bin we need an element of $\mathbf{1}$ which is consistent with $(1, *, 0)$, thus, we use $(1, 0, 0)$. For the fourth bin, we need an element of $\mathbf{0}$ which is consistent with $(*, *, 0)$. We can use either $(0, 0, 0)$ or $(1, 1, 0)$. W.l.o.g., we'll use $(1, 1, 0)$. We glue together $(0, 0, 0), (0, 1, v), (1, 0, 0), (1, 1, 0)$ to form $(0, 0, 0, 0, 1, v, 1, 0, 0, 1, 1, 0) \in (\mathbf{0}, \mathbf{v}, \mathbf{1}, \mathbf{0})$ which is also consistent with the subcube $(0, 0, 0, *, 1, *, 1, *, 0, *, *, 0)$.

To show that our Breaker's win pairing strategy can handle every 5-dimensional subcube, we consider all ways that the seven fixed coordinates can be distributed amongst the four bins. We partition these possibilites into the following categories: $(3, 3, 1, 0)$, $(3, 2, 2, 0)$, $(3, 2, 1, 1)$, $(2, 2, 2, 1)$, where, for example, $(0, 0, 0, *, 1, *, 1, *, 0, *, *, 0)$ is in category $(3, 2, 1, 1)$ because its first bin has 3 coordinates fixed, its second bin has 1 coordinate fixed, its third bin has 2 coordinates fixed, and its fourth bin has 1 coordinate fixed.

We now need to show that for any subcube in any of the four categories, we can find an edge in our Breaker's win pairing strategy that is consistent with that subcube. We will use certain properties possessed by the Breaker's win pairing strategy $PS(4, 2)$ (which carry over to the bin-forms in $\mathit{BinSetsPS}(4, 2)$) in order to help us prove that every subcube in a given category type is handled by one of the eight bin-forms in $\mathit{BinSetsPS}(4, 2)$.

Let us begin with $(2, 2, 2, 1)$. Suppose that S is in category $(2, 2, 2, 1)$ and that S has exactly 1 fixed coordinate in bin i. Observe that for each of the four coordinates, $PS(4, 2)$ has exactly two vectors with a v in that coordinate. We claim that either of the two bin-forms from $\mathit{BinSetsPS}(4, 2)$ which has the bin-set $\mathbf{v}$ in bin i will handle S. This is because $\mathbf{v}$ can handle any subcube from Q_3 with exactly 1 fixed coordinate. The other three bin-sets will each either be $\mathbf{0}$ or $\mathbf{1}$, thus, each of the other three bin-sets can handle any subcube from Q_3 with exactly 2 fixed coordinates. Thus, our eight bin-forms in $\mathit{BinSetsPS}(4, 2)$ can handle any subcube in category $(2, 2, 2, 1)$.

Now suppose that S is in category $(3, 3, 1, 0)$ and that S has all three coordinates fixed in bins i and j, where $i \neq j$. Let $S|_{\text{bin } k}$ be S restricted to bin k for $k \in [4]$, so that each of $S|_{\text{bin } i}$ and $S|_{\text{bin } j}$ is an element of $\{0, 1\}^3$. Because $PS(4, 2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4, 2)$, for each pair $\{i, j\} \in \binom{[4]}{2}$, and for each ordered pair $(b_i, b_j) \in \{0, 1\}^2$, there is a vector in $PS(4, 2)$ with value b_i in coordinate i and value b_j in coordinate j. Thus, there is a bin-form in $\mathit{BinSetsPS}(4, 2)$ whose bin-set in bin i matches the parity of $S|_{\text{bin } i}$ and whose bin-set in bin j matches the parity of $S|_{\text{bin } j}$. Since each of the bin-sets $\mathbf{0}, \mathbf{1}, \mathbf{v}$ can handle any subcube from Q_3 with 0 or 1 fixed coordinates, we can conclude that our eight bin-forms in $\mathit{BinSetsPS}(4, 2)$ can handle any subcube in category $(3, 3, 1, 0)$. As an example, suppose that $S = (*, *, *, 0, 1, 1, 0, 1, 0, *, 1, *)$. Bins 2 and 3 each have all three coordinates fixed. We have $(0, 1, 1) \in \mathbf{0}$ in bin 2 and $(0, 1, 0) \in \mathbf{1}$ in bin 3, thus, we search for a bin-form from $\mathit{BinSetsPS}(4, 2)$ of the form $(*, \mathbf{0}, \mathbf{1}, *)$. The fact that $PS(4, 2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4, 2)$ guarantees that we will find a bin-form from $\mathit{BinSetsPS}(4, 2)$ with the desired form. We note that $(\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{v})$ has the desired form and that it handles S.

We can use a similar argument to explain why our eight bin-forms in *BinSetsPS*$(4,2)$ can handle any subcube in category $(3,2,1,1)$. To handle the subcubes from Q_3 that have 3 fixed coordinates and 2 fixed coordinates, treat the bin with exactly 2 fixed coordinates as if it had 3 fixed coordinates and w.l.o.g., pretend those three fixed coordinates yielded an even parity vector. Then repeat the argument we used for $(3,3,1,0)$, which will work since the bin-sets $\mathbf{0}, \mathbf{1}, \mathbf{v}$ can handle any subcube from Q_3 with exactly 1 fixed coordinate. As an example, suppose that $S = (*,0,*,0,1,1,*,1,*,1,*,1)$. Bins 2 and 4 have three and two coordinates fixed, respectively. We have $(0,1,1) \in \mathbf{0}$ in bin 2, and $(1,*,1)$ in bin 4. We know that $(1,0,1) \in \mathbf{0}$ is consistent with $(1,*,1)$, so we pretend that we have $(1,0,1)$ in bin 4. Thus, we search for a bin-form from *BinSetsPS*$(4,2)$ of the form $(*,\mathbf{0},*,\mathbf{0})$. The fact that $PS(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$ guarantees that we will find a bin-form from *BinSetsPS*$(4,2)$ with the desired form. We note that $(\mathbf{v},\mathbf{0},\mathbf{0},\mathbf{0})$ has the desired form and that it handles S.

Suppose that S is in category $(3,2,2,0)$ and that S has zero coordinates fixed in bin i and all three coordinates fixed in bin j. Recall that for each of the four coordinates, $PS(4,2)$ has exactly two vectors with a v in that coordinate and that those two vectors are complements of each other (where we consider v to be its own complement). Thus, there are two bin-forms in *BinSetsPS*$(4,2)$ that have a $\mathbf{v}$ in bin i, and exactly one of those bin-forms has a bin-set whose parity equals the parity of $S|_{\text{bin } j}$. We select that bin-form to handle S. Each of the other two bin-sets will be either $\mathbf{0}$ or $\mathbf{1}$, thus, each of the other two bin-sets can handle any subcube from Q_3 with exactly 2 fixed coordinates. Thus, our eight bin-forms in *BinSetsPS*$(4,2)$ can handle any subcube in category $(3,2,2,0)$. As an example, suppose that $S = (1,0,*,0,1,1,*,*,*,1,*,1)$. Bin 3 has zero coordinates fixed, so we begin by searching for a bin-form from *BinSetsPS*$(4,2)$ of the form $(*,*,\mathbf{v},*)$. Both $(\mathbf{0},\mathbf{0},\mathbf{v},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{v},\mathbf{0})$ have the desired form, but only $(\mathbf{0},\mathbf{0},\mathbf{v},\mathbf{1})$ is consistent with S in bin 2, because $S|_{\text{bin } 2} = (0,1,1) \in \mathbf{0}$. We note that $(\mathbf{0},\mathbf{0},\mathbf{v},\mathbf{1})$ handles S.

We have shown that *BinPS*$(4,2)$ handles every 5-dimensional subcube of Q_{12}, but we also have to prove that *BinPS*$(4,2)$ is a matching. To see this, we use the fact that $PS(4,2)$ is a matching. For any two distinct edges in a matching, there is at least one coordinate that is fixed by both edges in which one of the edges has a 0 and the other has a 1, i.e., there is some *mutually-non-v* coordinate where the two edges differ. Otherwise, there would be a vertex that is shared by two distinct edges of the matching. For example, $(v,0,1,0)$ and $(1,v,1,0)$ do not have such a coordinate, and the vertex $(1,0,1,0)$ is in both edges.

We need to show that for any two distinct edges in *BinPS*$(4,2)$, there is some mutually-non-v coordinate where they differ. It should be clear that any two distinct edges that belong to the same bin-form of *BinSetsPS*$(4,2)$ will satisfy this property, e.g., any two distinct edges from $(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v})$ will differ in a mutually-non-v coordinate.

Now suppose we take two distinct edges that are contained in different bin-forms of *BinSetsPS*$(4,2)$, e.g., an edge from $(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v})$ and an edge from $(\mathbf{0},\mathbf{v},\mathbf{1},\mathbf{0})$. (Let us refer to the two distinct edges as *Q_{12}-edges* to distinguish them from edges in $PS(4,2)$ which we will call *Q_4-edges*.) Since *BinSetsPS*$(4,2)$ is derived from $PS(4,2)$, there are two distinct Q_4-edges from $PS(4,2)$ that give us our two different bin-forms from *BinSetsPS*$(4,2)$. Since $PS(4,2)$ is a matching, there is a mutually-non-v coordinate in which those two distinct Q_4-edges from $PS(4,2)$ differ. Thus, the two different bin-forms from *BinSetsPS*$(4,2)$, which contain our two Q_{12}-edges, differ in the bin that corresponds to that coordinate, i.e., one bin-form has bin-set $\mathbf{0}$ in that bin and the other bin-form has bin-set $\mathbf{1}$ in that bin. Since $\mathbf{0} \cap \mathbf{1} = \emptyset$, the two distinct Q_{12}-edges differ in at least one coordinate which is contained in that same "mutually-non-$\mathbf{v}$" bin where the two bin-forms from $BinSetsPS(4,2)$ differ. E.g., because $(0,1,0,v)$ has 0 in the third coordinate and $(0,v,1,0)$ has 1 in the third coordinate, then $(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v})$ has $\mathbf{0}$ in the third bin and $(\mathbf{0},\mathbf{v},\mathbf{1},\mathbf{0})$ has $\mathbf{1}$ in the third bin. Thus, every edge in $(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v})$ has an even parity vector in its third bin and every edge in $(\mathbf{0},\mathbf{v},\mathbf{1},\mathbf{0})$ has an odd parity vector in its third bin. Thus, an edge from $(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v})$ and an edge from $(\mathbf{0},\mathbf{v},\mathbf{1},\mathbf{0})$ will differ in at least one mutually-non-v coordinate within bin 3.

These arguments generalize to any $BinPS(4,2)$. Thus, we have the following lemma.

Lemma 1 *Using any bin size n, the set of edges $BinPS(4,2)$ is a matching, therefore, it can be used as a pairing strategy for a game played on the vertices of Q_{4n}.*

3 Expanding a Fixed Pairing Strategy

Now that we know $BinPS(4,2)$ is a matching for any bin size n, we would like to know what size subcubes $BinPS(4,2)$ can handle, based on the Breaker's win pairing strategy available for a board of dimension n. This leads us to the following theorem.

Theorem 1 *If there exists a Breaker's win pairing strategy for $\mathcal{Q}(n,k)$, then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4n,b)$, where $b = \max\{4k-3, n+k\}$.*

Proof of Theorem 1: Similar to the $\mathcal{Q}(12,5)$ example, we partition the $4n$ coordinates into four bins of size n, so that bin j contains coordinates $1+(j-1)n$ through jn for $j \in [4]$. Suppose that a subcube of Q_{4n} has exactly c fixed coordinates in a given bin. If $c = n$, then we say that the bin is *full* or a *full-bin*; if $n-k+1 \leq c \leq n-1$, then we say that the bin is *heavy* or a *heavy-bin*; if $0 \leq c \leq n-k$, then we say that the bin is *light* or a *light-bin*. If a bin is not full, i.e., it is heavy or light, then we say it is a *non-full-bin*.

We will use $BinPS(4,2)$, which is the union of the bin-forms in $BinSetsPS(4,2)$, as our Breaker's win pairing strategy. In this version of $BinSetsPS(4,2)$, we have that $\mathbf{0}$ is the set of even parity vectors from $\{0,1\}^n$, that $\mathbf{1}$ is the set of odd parity vectors from $\{0,1\}^n$, and that $\mathbf{v}$ is a Breaker's win pairing strategy for $\mathcal{Q}(n,k)$. Similar to the $\mathcal{Q}(12,5)$ example, we note that $\mathbf{0}$ and $\mathbf{1}$ can each handle any subcube of Q_n with dimension at least 1, i.e., they can each handle any non-full-bin. Since $\mathbf{v}$ is a Breaker's win pairing strategy for $\mathcal{Q}(n,k)$, it can handle any light bin.

Let S be a b-dimensional subcube of Q_{4n}. Then there are $4n-b$ fixed coordinates in S. Since $4n-b \leq 4n-4k+3$, S has at least one light-bin. Since $4n-b \leq 3n-k$ and $k \geq 1$, S has at most two full-bins.

First suppose that S has no full-bins, and that bin i of S is light. Let $A \in BinSetsPS(4,2)$ be such that $\mathbf{v}$ is in bin i of A. Since bin i of S is light, $\mathbf{v}$ can handle $S|_{\text{bin } i}$. In each of the other bins, A has either a $\mathbf{0}$ or a $\mathbf{1}$. Since each of $\mathbf{0}$ and $\mathbf{1}$ can handle any non-full-bin, A handles S.

Suppose instead that S has exactly two full-bins, namely, bin i and bin j. Since $PS(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$, there is a bin-form $A \in BinSetsPS(4,2)$ whose bin-set in bin i matches the parity of $S|_{\text{bin } i}$ and whose bin-set in bin j matches the parity of $S|_{\text{bin } j}$. There are $2n - b$ other fixed coordinates in S. Since $b \geq n + k$, we have that $2n - b \leq n - k$. Thus, the other two bins must be light, and they can each be handled by any bin-set $\mathbf{0}, \mathbf{1}$, or $\mathbf{v}$. Therefore, A handles S.

Suppose that S has exactly one full-bin. Since S also has a light-bin, suppose bin i is light and bin j is full. Recall that $\mathit{BinSetsPS}(4,2)$ contains exactly two bin-forms with $\mathbf{v}$ in bin i, and exactly one of those bin-forms has a bin-set in bin j whose parity matches $S|_{\text{bin } j}$. Let $A \in \mathit{BinSetsPS}(4,2)$ be that bin-form. In each of the other two bins, A has either a $\mathbf{0}$ or a $\mathbf{1}$. Since each of $\mathbf{0}$ and $\mathbf{1}$ can handle any non-full-bin (and the other two bins are non-full-bins), A handles S.

In light of Lemma 1, we conclude that $\mathit{BinPS}(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4n, b)$. □

Corollary 1 *Suppose there exists a Breaker's win pairing strategy for $\mathcal{Q}(n,k)$.*

(a) If $k \geq n/3 + 1$, then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4n, 4k-3)$.

(b) If $k = \lfloor n/3 \rfloor + 1$, then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4n, \lfloor 4n/3 \rfloor + 1)$.

4 Rotating the Pairing Strategies

In Theorem 1, the number of fixed coordinates was basically bounded by avoiding a configuration with two full-bins and one heavy-bin or a configuration with four heavy-bins, i.e., a $(F, F, H, *)$ configuration or a (H, H, H, H) configuration. Thus, in Theorem 1, we needed the number of fixed coordinates to be at most $\min\{2n + n - k, 4(n - k + 1) - 1\}$.

To see why we wish to avoid a $(F, F, H, *)$ configuration, let us return to the Q_{12} example. Suppose that we tried to block all 4-dimensional subcubes of Q_{12}, which corresponds to fixing 8 coordinates. When fixing 8 coordinates, we encounter the category $(3,3,2,0)$, which is a $(F, F, H, *)$ configuration. Let us describe a 4-dimensional subcube which is not handled by $\mathit{BinPS}(4,2)$. Let $S = (*, 1, 1, 0, 0, 0, 0, 0, 0, *, *, *)$, which we partition into $(*, 1, 1)$, $(0,0,0)$, $(0,0,0)$, $(*, *, *)$. Bins 2 and 3 both need to use bin-set $\mathbf{0}$. Thus, we need a bin-form from $\mathit{BinSetsPS}(4,2)$ of the form $(*, \mathbf{0}, \mathbf{0}, *)$. However, for each 2-dimensional subcube, $\mathit{PS}(4,2)$ contains exactly one edge that is consistent with that subcube. Thus, only $(\mathbf{v}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ has the form $(*, \mathbf{0}, \mathbf{0}, *)$. Naturally, we chose S so that its heavy-bin (two fixed coordinates) is in the first bin which contains $\mathbf{v}$, since $\mathbf{v}$ cannot handle every subcube with two fixed coordinates; and of course, we chose a 1-dimensional subcube of Q_3 which is not handled by $\mathbf{v} = \{(v, 0, 1), (1, v, 0), (0, 1, v)\}$ to be in the first bin of S. Thus, there is no edge in $\mathit{BinPS}(4,2)$ that handles S.

We can give a similar argument to explain why we need to avoid a (H, H, H, H) configuration. For example, if we let $S = (*, 1, 1, *, 1, 1, *, 1, 1, *, 1, 1)$, then every bin contains the 1-dimensional subcube $(*, 1, 1)$ which is not handled by $\mathbf{v}$. Thus, no matter which edge we use from $\mathit{BinPS}(4,2)$, our edge will not be consistent with $(*, 1, 1)$ in the bin where our edge uses a vector from $\mathbf{v}$. These ideas, of course, generalize to other examples of $\mathit{BinPS}(4,2)$ where our bins have size other than 3. Essentially, $\mathit{BinPS}(4,2)$ cannot handle a situation where the bin with $\mathbf{v}$ is forced to be a heavy-bin.

However, there is the promise of a workaround for the $(F, F, H, *)$ case. The basic idea is to have different versions of $\mathbf{v}$ available which we can rotate through as we glue together the n-dimensional vectors to form a $4n$-dimensional vector.

To illustrate the idea, let us explain how we can construct a Breaker's win pairing strategy for $\mathcal{Q}(9,4)$ using the rotating pairing strategy idea. For simplicity, we have chosen an example that is based on $\mathbf{v}_3 = \{(v,0,1),(1,v,0),(0,1,v)\}$ and has only three bins, each of size three. We will still use $\mathbf{0}_3$ and $\mathbf{1}_3$. However, we will use four different Breaker's win pairing strategies for $\mathcal{Q}(3,2)$ that are all distinct from $\mathbf{v}_3$. Let $\mathbf{v}^{(0)} = \{(v,0,0),(1,v,1),(0,1,v)\}$, let $\mathbf{v}^{(1)} = \{(v,0,1),(0,v,0),(1,1,v)\}$, let $\mathbf{v}^{(2)} = \{(v,1,0),(0,v,1),(1,0,v)\}$, and let $\mathbf{v}^{(3)} = \{(v,1,1),(1,v,0),(0,0,v)\}$. In $BinPS(4,2)$, we had to avoid forcing a heavy-bin into the bin containing $\mathbf{v}$ because $\mathbf{v}$ could not handle up to $n-1$ fixed coordinates, like $\mathbf{0}$ and $\mathbf{1}$ can. However, since $\mathbf{v}^{(0)} \cup \mathbf{v}^{(1)} \cup \mathbf{v}^{(2)} \cup \mathbf{v}^{(3)}$ contains every edge in Q_3, if we appropriately rotate through those Breaker's win pairing strategies we will be able to handle a heavy-bin in the *v-bin*, i.e., the bin containing a Breaker's win pairing strategy for $\mathcal{Q}(3,2)$.

Let us describe one way to rotate through those four Breaker's win pairing strategies. If we did a straight analogue of $BinPS(4,2)$, we would use the three bin-forms $(\mathbf{v},\mathbf{0},\mathbf{1})$, $(\mathbf{1},\mathbf{v},\mathbf{0})$, $(\mathbf{0},\mathbf{1},\mathbf{v})$ to build our Breaker's win pairing strategy. We will use edges similar to those found in these three bin-forms, except our edges will use elements from one of $\mathbf{v}^{(0)}$, $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$, $\mathbf{v}^{(3)}$ in their v-bin. To achieve this, index the elements of $\mathbf{0}$ and the elements of $\mathbf{1}$ using $\{0,1,2,3\}$. We could do this in increasing order where each element is viewed as a base 2 representation of a number. For example, $\mathbf{0}(0) = (0,0,0)$, $\mathbf{0}(1) = (0,1,1)$, $\mathbf{0}(2) = (1,0,1)$, $\mathbf{0}(3) = (1,1,0)$. Thus, $\mathbf{0}(j)$ is the jth element of $\mathbf{0}$ according to our fixed indexing. (In the proof of Theorem 2, we use partitions of $\mathbf{0}$ and $\mathbf{1}$ to determine the indices of the elements in those sets.) We will replace the bin-forms $(\mathbf{v},\mathbf{0},\mathbf{1})$, $(\mathbf{1},\mathbf{v},\mathbf{0})$, $(\mathbf{0},\mathbf{1},\mathbf{v})$ with the bin-forms $(\mathbf{v}^R,\mathbf{0},\mathbf{1})$, $(\mathbf{1},\mathbf{v}^R,\mathbf{0})$, $(\mathbf{0},\mathbf{1},\mathbf{v}^R)$ which are defined as follows. For example, $(\mathbf{0},\mathbf{1},\mathbf{v}^R)$ contains 9-dimensional vectors, where the first bin of each vector contains an element of $\mathbf{0}$, the second bin of each vector contains an element of $\mathbf{1}$, and the third bin of each vector contains an element of some $\mathbf{v}^{(j)}$. We determine which bin-set $\mathbf{v}^{(j)}$ from the indices of the two vectors from $\mathbf{0}$ and $\mathbf{1}$ that are in the edge. Specifically, each edge in $(\mathbf{0},\mathbf{1},\mathbf{v}^R)$ will have the form $(\vec{x},\vec{y},\vec{z})$, where $\vec{x} \in \mathbf{0}$ and $\vec{y} \in \mathbf{1}$ and $\vec{z} \in \mathbf{v}^{(j)}$, where j is equal to the sum of the indices of $\vec{x}$ and $\vec{y}$ modulo 4. For example, $\{(0,1,1,1,0,0,\vec{z}) : \vec{z} \in \mathbf{v}^{(3)}\}$ is a subset of $(\mathbf{0},\mathbf{1},\mathbf{v}^R)$ because the index of $(0,1,1)$ is 1 and the index of $(1,0,0)$ is 2. (Again, when we write $(0,1,1,1,0,0,\vec{z})$, we really mean the 9-dimensional vector that corresponds to that expression.) To help us write a concise definition of $(\mathbf{0},\mathbf{1},\mathbf{v}^R)$, let $\mathrm{Index}(\vec{x})$ equal the index of $\vec{x}$, for $\vec{x} \in \mathbf{0}$ or $\vec{x} \in \mathbf{1}$. For example, $\mathrm{Index}((0,1,1)) = 1$ and $\mathrm{Index}((1,0,0)) = 2$. Thus,

$$(\mathbf{0},\mathbf{1},\mathbf{v}^R) = \{(\vec{x},\vec{y},\vec{z}) : \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{1}, \vec{z} \in \mathbf{v}^{(j)}, \text{where } j = (\mathrm{Index}(\vec{x}) + \mathrm{Index}(\vec{y})) \bmod 4\}.$$

We define $(\mathbf{v}^R,\mathbf{0},\mathbf{1})$ and $(\mathbf{1},\mathbf{v}^R,\mathbf{0})$ similarly.

To show that $(\mathbf{v}^R,\mathbf{0},\mathbf{1}) \cup (\mathbf{1},\mathbf{v}^R,\mathbf{0}) \cup (\mathbf{0},\mathbf{1},\mathbf{v}^R)$ can handle every 4-dimensional subcube S, we show that every category of five fixed coordinates can be handled, i.e., $(3,2,0)$, $(3,1,1)$, and $(2,2,1)$.

Suppose S is in category $(2,2,1)$ and that S has exactly 1 fixed coordinate in bin i. We will choose the bin-form with $\mathbf{v}^R$ in bin i. The other two bins contain bin-sets $\mathbf{0}$ and $\mathbf{1}$, which can each handle any subcube with 2 fixed coordinates. Choose $\vec{x} \in \mathbf{0}$ and $\vec{y} \in \mathbf{1}$ which are consistent with S in the other two bins. Suppose $j = (\text{Index}(\vec{x})+\text{Index}(\vec{y})) \bmod 4$. Since $\mathbf{v}^{(j)}$ is a Breaker's win pairing strategy for $\mathcal{Q}(3,2)$, we can find $\vec{z} \in \mathbf{v}^{(j)}$ that is consistent with S in bin i. The edge which is formed by gluing together $\vec{x}, \vec{y}$ and $\vec{z}$ (with each 3-dimensional vector in the correct bin) is an edge in our Breaker's win pairing strategy and handles S.

Suppose S is in category $(3,1,1)$ and S has three coordinates fixed in bin i. Observe that $S|_{\text{bin } i}$ is a vertex in Q_3. Choose the bin-form from $(\mathbf{v}^R, \mathbf{0}, \mathbf{1})$, $(\mathbf{1}, \mathbf{v}^R, \mathbf{0})$, $(\mathbf{0}, \mathbf{1}, \mathbf{v}^R)$ whose bin-set in bin i matches the parity of $S|_{\text{bin } i}$. The other two bins of S have only 1 coordinate fixed. Since $\mathbf{v}^{(j)}, \mathbf{0}, \mathbf{1}$ can each handle any subcube of Q_3 with 1 fixed coordinate, there is an edge in our Breaker's win pairing strategy that handles S.

Suppose S is in category $(3,2,0)$ and S has three coordinates fixed in bin i, and two coordinates fixed in bin k. Thus, $S|_{\text{bin } i}$ is a vertex in Q_3 and $S|_{\text{bin } k}$ is an edge in Q_3. Again, choose the bin-form from $(\mathbf{v}^R, \mathbf{0}, \mathbf{1})$, $(\mathbf{1}, \mathbf{v}^R, \mathbf{0})$, $(\mathbf{0}, \mathbf{1}, \mathbf{v}^R)$ whose bin-set ($\mathbf{0}$ or $\mathbf{1}$) in bin i matches the parity of $S|_{\text{bin } i}$. If this bin-form has bin-set $\mathbf{0}$ or $\mathbf{1}$ in bin k, then our Breaker's win pairing strategy can handle S, since $\mathbf{0}$ and $\mathbf{1}$ can each handle subcubes of Q_3 with 2 fixed coordinates. However, if the bin-form has $\mathbf{v}^R$ in bin k, then we have to more carefully find an edge that handles S. Since $S|_{\text{bin } k}$ is an edge in Q_3 and $\mathbf{v}^{(0)} \cup \mathbf{v}^{(1)} \cup \mathbf{v}^{(2)} \cup \mathbf{v}^{(3)}$ contains every edge in Q_3, there is a Breaker's win pairing strategy $\mathbf{v}^{(j)}$ that contains $S|_{\text{bin } k}$. In the bin with no fixed coordinates, we choose an element $\vec{x}$ of $\mathbf{0}$ or $\mathbf{1}$ (whichever the case happens to be) so that $(\text{Index}(S|_{\text{bin } i}) + \text{Index}(\vec{x})) \bmod 4 = j$. We can choose such a vector $\vec{x}$ because S has no fixed coordinates in the bin containing $\vec{x}$, thus, any $\vec{x}$ is consistent with S in that bin. Therefore, our Breaker's win pairing strategy can handle S. As an example, suppose that $S = (1,0,*,1,0,1,*,*,*)$. Bins 1 and 2 have two and three coordinates fixed, respectively. We have $(1,0,1) \in \mathbf{0}$ in bin 2, thus, we must find an edge in $(\mathbf{v}^R, \mathbf{0}, \mathbf{1})$ to handle S. We have $(1,0,*)$ in bin 1, thus, we need to use a Breaker's win pairing strategy $\mathbf{v}^{(j)}$ which contains edge $(1,0,v)$. We observe that $(1,0,v) \in \mathbf{v}^{(2)}$. We have $(*,*,*)$ in bin 3, so any $\vec{x} \in \mathbf{1}$ will be consistent with S in bin 3. However, we must choose $\vec{x}$ so that $(\text{Index}(\vec{x}) + \text{Index}((1,0,1))) \bmod 4 = 2$. Since $\text{Index}((1,0,1)) = 2$, we choose $\vec{x} = (0,0,1)$ because $\text{Index}((0,0,1)) = 0$. Thus, $(1,0,v,1,0,1,0,0,1) \in (\mathbf{v}^R, \mathbf{0}, \mathbf{1})$ handles S.

Now that we have presented an example which illustrates the idea behind rotating pairing strategies, let us define our new Breaker's win pairing strategy in general. Suppose we have a set of matchings $\{\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(m)}\}$ such that each $\mathbf{v}^{(j)}$ is a Breaker's win pairing strategy for $\mathcal{Q}(n,k)$ and $\bigcup_j \mathbf{v}^{(j)}$ equals the set of edges of Q_n. Also suppose that we partition $\mathbf{0}$ into $\mathbf{0}^{(0)}, \ldots, \mathbf{0}^{(m-1)}$ and $\mathbf{1}$ into $\mathbf{1}^{(0)}, \ldots, \mathbf{1}^{(m-1)}$. For $\vec{x} \in \mathbf{0}^{(j)}$ or $\vec{x} \in \mathbf{1}^{(j)}$, let $\text{Index}(\vec{x}) = j$, i.e., the index of the set which contains $\vec{x}$. Let

$$\begin{aligned} \mathit{BinSetsPS}^R(4,2) = \{&(\mathbf{v}^R, \mathbf{0}, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \mathbf{v}^R, \mathbf{1}, \mathbf{0}), (\mathbf{0}, \mathbf{0}, \mathbf{v}^R, \mathbf{1}), (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{v}^R), \\ &(\mathbf{v}^R, \mathbf{1}, \mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{v}^R, \mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{1}, \mathbf{v}^R, \mathbf{0}), (\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{v}^R)\}, \end{aligned}$$

where, for example,

$$\begin{aligned} (\mathbf{0}, \mathbf{0}, \mathbf{v}^R, \mathbf{1}) = \{(\vec{x}, \vec{y}, \vec{z}, \vec{w}) : \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{0}, \vec{z} \in \mathbf{v}^{(j)}, \vec{w} \in \mathbf{1}, \\ \text{where } j = (\text{Index}(\vec{x}) + \text{Index}(\vec{y}) + \text{Index}(\vec{w})) \bmod m\}. \end{aligned}$$

Let

$$\mathit{BinPS}^R(4,2) = (\mathbf{v}^R, \mathbf{0}, \mathbf{0}, \mathbf{0}) \cup (\mathbf{0}, \mathbf{v}^R, \mathbf{1}, \mathbf{0}) \cup (\mathbf{0}, \mathbf{0}, \mathbf{v}^R, \mathbf{1}) \cup (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{v}^R)$$

$$\cup\, (\mathbf{v}^R, \mathbf{1}, \mathbf{1}, \mathbf{1}) \cup (\mathbf{1}, \mathbf{v}^R, \mathbf{0}, \mathbf{1}) \cup (\mathbf{1}, \mathbf{1}, \mathbf{v}^R, \mathbf{0}) \cup (\mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{v}^R).$$

We now prove the following theorem.

Theorem 2 *Suppose there exists a set of matchings $\{\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(m-1)}\}$ such that each $\mathbf{v}^{(j)}$ is a Breaker's win pairing strategy for $\mathcal{Q}(n,k)$ and $\bigcup_j \mathbf{v}^{(j)}$ equals the set of edges of Q_n. Moreover, suppose that there is a partition of $\mathbf{0}_n$ (and a partition of $\mathbf{1}_n$) of size m such that every subcube of Q_n of dimension $n-k+2$ contains at least one vertex from each of the sets in the partition. Then there exists a Breaker's win pairing strategy for $\mathcal{Q}(4n, b)$, where $b = \max\{4k-3, n+1\}$.*

Proof of Theorem 2: We partition the $4n$ coordinates into four bins and define *full-bins*, *heavy-bins*, *light-bins*, $\mathbf{0}$ and $\mathbf{1}$ exactly as we did in the proof of Theorem 1.

Let $\mathbf{0}^{(0)}, \ldots, \mathbf{0}^{(m-1)}$ and $\mathbf{1}^{(0)}, \ldots, \mathbf{1}^{(m-1)}$ be the partitions of $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively, where every subcube of Q_n of dimension $n-k+2$ contains at least one vertex from $\mathbf{0}^{(j)}$ and at least one vertex from $\mathbf{1}^{(j)}$ for $0 \le j \le m-1$. For $\vec{x} \in \mathbf{0}^{(j)}$ or $\vec{x} \in \mathbf{1}^{(j)}$, let $\mathrm{Index}(\vec{x}) = j$. We use $BinPS^R(4,2)$ as our Breaker's win pairing strategy.

Let $b = \max\{4k-3, n+1\}$, and let S be a b-dimensional subcube of Q_{4n}. Then there are $4n-b$ fixed coordinates in S. Since $4n-b \le 4n-4k+3$, there is at least one light bin. Since $4n-b \le 4n-(n+1) = 3n-1$, there are at most two full-bins.

We can follow the proof of Theorem 1 for the cases when S has no full-bins or S has exactly one full-bin.

Suppose that S has exactly two full-bins, namely, bin i_1 and bin i_2. Thus, $S|_{\text{bin } i_1}$ and $S|_{\text{bin } i_2}$ are both vertices in Q_n. Since $PS(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$, there is a bin-form $A \in BinSetsPS^R(4,2)$ whose bin-set ($\mathbf{0}$ or $\mathbf{1}$) in bin i_1 matches the parity of $S|_{\text{bin } i_1}$ and whose bin-set ($\mathbf{0}$ or $\mathbf{1}$) in bin i_2 matches the parity of $S|_{\text{bin } i_2}$. There are at most $n-1$ other fixed coordinates in S. If both of the other two remaining bins are light, then they can each be handled by any bin-set $\mathbf{0}, \mathbf{1}$, or $\mathbf{v}^{(j)}$. Suppose instead that one bin, say bin i_3, is heavy and the other, bin i_4, is light. If the bin-form A has a $\mathbf{0}$ or $\mathbf{1}$ in bin i_3, then A handles S. Suppose instead that A has $\mathbf{v}^R$ in bin i_3, and w.l.o.g., A has $\mathbf{0}$ in bin i_4. Since $\bigcup_j \mathbf{v}^{(j)}$ equals the set of edges of Q_n and $S|_{\text{bin } i_3}$ has at most $n-1$ fixed coordinates, there is a bin-set $\mathbf{v}^{(j)}$ which contains an edge that can handle $S|_{\text{bin } i_3}$. Let $c \in \{0, \ldots, m-1\}$ satisfy $(\mathrm{Index}(S|_{\text{bin } i_1}) + \mathrm{Index}(S|_{\text{bin } i_2}) + c) \bmod m = j$. Since every subcube of Q_n of dimension $n-k+2$ contains at least one vertex from $\mathbf{0}^{(c)}$, there is a vertex $\vec{x} \in \mathbf{0}^{(c)}$ which handles $S|_{\text{bin } i_4}$ as long as $S|_{\text{bin } i_4}$ has at most $k-2$ fixed coordinates. Since bin i_3 is heavy and there are at most $n-1$ fixed coordinates distributed between bin i_3 and bin i_4, $S|_{\text{bin } i_4}$ has at most $k-2$ fixed coordinates. Therefore, we can find an edge in A that handles S. $\square$

For Theorem 2 to be useful, we need to exhibit a set of Breaker's win pairing strategies for $\mathcal{Q}(n,k)$ whose union is the set of edges of Q_n where $k < n/3+1$. Otherwise, there will be no improvement over the results obtained from Theorem 1. We will use Theorem 2 to prove that for $d \ge 0$, there is a Breaker's win pairing strategy for $\mathcal{Q}(4^{d+1}, 4^d+1)$.

We have already established the case $d=0$ via $PS(4,2)$. To establish the case $d=1$, we will need to introduce the following Breaker's win pairing strategies:

$$PS_0(4,2) = \{(v,0,0,0), (0,v,1,0), (0,0,v,1), (0,1,0,v),$$

$$(v,1,1,1),(1,v,0,1),(1,1,v,0),(1,0,1,v)\},$$

$$PS_1(4,2) = \{(v,0,1,1),(0,v,0,1),(0,0,v,0),(0,1,1,v),$$
$$(v,1,0,0),(1,v,1,0),(1,1,v,1),(1,0,0,v)\},$$

$$PS_2(4,2) = \{(v,1,0,1),(0,v,1,1),(0,1,v,0),(0,0,0,v),$$
$$(v,0,1,0),(1,v,0,0),(1,0,v,1),(1,1,1,v)\},$$

$$PS_3(4,2) = \{(v,1,1,0),(0,v,0,0),(0,1,v,1),(0,0,1,v),$$
$$(v,0,0,1),(1,v,1,1),(1,0,v,0),(1,1,0,v)\}.$$

One can check that these matchings partition the set of edges of Q_4, and each $PS_j(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4,2)$ for $0 \le j \le 3$. We can let $\mathbf{v}^{(j)} = PS_j(4,2)$ for $0 \le j \le 3$, then apply Theorem 2 to obtain a Breaker's win pairing strategy for $\mathcal{Q}(16,5)$, i.e., case $d=1$. However, to establish the case $d=2$, we will need 16 Breaker's win pairing strategies for $\mathcal{Q}(16,5)$ whose union is the set of edges of Q_{16}. To do this, we define the following sets of bin-forms based on $PS_j(4,2)$ for $0 \le j \le 3$. For example, when $j=0$, let

$$\mathit{BinSetsPS}_0^{R(s)}(4,2) = \{(\mathbf{v}^{R(s)},\mathbf{0},\mathbf{0},\mathbf{0}),(\mathbf{0},\mathbf{v}^{R(s)},\mathbf{1},\mathbf{0}),(\mathbf{0},\mathbf{0},\mathbf{v}^{R(s)},\mathbf{1}),(\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v}^{R(s)}),$$
$$(\mathbf{v}^{R(s)},\mathbf{1},\mathbf{1},\mathbf{1}),(\mathbf{1},\mathbf{v}^{R(s)},\mathbf{0},\mathbf{1}),(\mathbf{1},\mathbf{1},\mathbf{v}^{R(s)},\mathbf{0}),(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{v}^{R(s)})\},$$

and

$$\mathit{BinPS}_0^{R(s)}(4,2) = (\mathbf{v}^{R(s)},\mathbf{0},\mathbf{0},\mathbf{0}) \cup (\mathbf{0},\mathbf{v}^{R(s)},\mathbf{1},\mathbf{0}) \cup (\mathbf{0},\mathbf{0},\mathbf{v}^{R(s)},\mathbf{1}) \cup (\mathbf{0},\mathbf{1},\mathbf{0},\mathbf{v}^{R(s)})$$
$$\cup (\mathbf{v}^{R(s)},\mathbf{1},\mathbf{1},\mathbf{1}) \cup (\mathbf{1},\mathbf{v}^{R(s)},\mathbf{0},\mathbf{1}) \cup (\mathbf{1},\mathbf{1},\mathbf{v}^{R(s)},\mathbf{0}) \cup (\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{v}^{R(s)}),$$

where, for example,

$$(\mathbf{0},\mathbf{0},\mathbf{v}^{R(s)},\mathbf{1}) = \{(\vec{x},\vec{y},\vec{z},\vec{w}) : \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{0}, \vec{z} \in \mathbf{v}^{(j)}, \vec{w} \in \mathbf{1},$$
$$\text{where } j = (s + \mathrm{Index}(\vec{x}) + \mathrm{Index}(\vec{y}) + \mathrm{Index}(\vec{w})) \bmod m\},$$

so that the indices used in the v-bin are "shifted by s." Note that when $s=0$, we obtain $(\mathbf{0},\mathbf{0},\mathbf{v}^{R(0)},\mathbf{1}) = (\mathbf{0},\mathbf{0},\mathbf{v}^{R},\mathbf{1})$. We claim that $\mathit{BinPS}_j^{R(s)}(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(16,5)$ for $0 \le j \le 3$ and $0 \le s \le 3$, and that the Breaker's win pairing strategies $\mathit{BinSetsPS}_j^{R(s)}(4,2)$ form a partition of the edges of Q_{16} if we use $\mathbf{0}_4$, $\mathbf{1}_4$, and $\mathbf{v}^{(j)} = PS_j(4,2)$ in the definitions of our bin-forms. However, at this point let us state and prove the theorem in general.

Theorem 3 *For each $d \ge 0$, there exist 4^{d+1} disjoint Breaker's win pairing strategies for $\mathcal{Q}(4^{d+1}, 4^d+1)$ with equal cardinalities which partition the set of edges of $Q_{4^{d+1}}$.*

Proof of Theorem 3: We proceed by induction on d. The Breaker's win pairing strategies $PS_j(4,2)$ for $0 \le j \le 3$ handle the case $d=0$. Let $d \ge 1$. By the inductive hypothesis, there exist 4^d disjoint Breaker's win pairing strategies $\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(4^d-1)}$ for $\mathcal{Q}(4^d, 4^{d-1}+1)$ with equal cardinalities which partition the set of edges of Q_{4^d}. We will show that $BinPS_j^{R(s)}(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4^{d+1}, 4^d+1)$ for $0 \le j \le 3$ and $0 \le s \le 4^d - 1$, where we use $\mathbf{0}_{4^d}$, $\mathbf{1}_{4^d}$, and the Breaker's win pairing strategies $\mathbf{v}^{(j)}$ from the inductive hypothesis in the definitions of the bin-forms. Moreover, we will show that the Breaker's win pairing strategies $BinPS_j^{R(s)}(4,2)$ form a partition of the edges of $Q_{4^{d+1}}$.

We will apply a few minor modifications to the proof of Theorem 2 in order to see that $BinPS_j^{R(s)}(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4^{d+1}, 4^d+1)$. Let $n = m = 4^d$ and $k = 4^{d-1}+1$. We use $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(4^d-1)}$ as the Breaker's win pairing strategies for $\mathcal{Q}(n,k)$. We apply Lemma 2 (see below) to conclude that there is a partition of $\mathbf{0}_{4^d}$ (and a partition of $\mathbf{1}_{4^d}$) of size m such that every subcube of Q_n of dimension $\frac{1}{2}4^d+1$ contains at least one vertex from each of the sets in the partition. Since $n-k+2 = \frac{3}{4}4^d+1 \ge \frac{1}{2}4^d+1$, the hypotheses for Theorem 2 are satisfied. We can substitute $BinPS_j^{R(s)}(4,2)$ for $BinPS^R(4,2)$ throughout the proof and reach the conclusion that $BinPS_j^{R(s)}(4,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(4^{d+1}, 4^d+1)$. The only minor change we make is to state, "let $c \in \{0, \dots, m-1\}$ satisfy $(s + \mathrm{Index}(S|_{\mathrm{bin}\ i_1}) + \mathrm{Index}(S|_{\mathrm{bin}\ i_2}) + c) \bmod m = j$."

Let $E(Q_{4^{d+1}})$ be the set of edges of $Q_{4^{d+1}}$. We will show that

$$E(Q_{4^{d+1}}) \subseteq \bigcup_{j,s} BinPS_j^{R(s)}(4,2),$$

which implies $\bigcup_{j,s} BinPS_j^{R(s)}(4,2) = E(Q_{4^{d+1}})$.

Let $S \in E(Q_{4^{d+1}})$. Suppose that $S|_{\mathrm{bin}\ i_1}$, $S|_{\mathrm{bin}\ i_2}$, $S|_{\mathrm{bin}\ i_3}$ are all vertices in Q_{4^d} and $S|_{\mathrm{bin}\ i_4}$ is an edge in Q_{4^d}. Let $\vec{x}$ be the edge in Q_4 which satisfies the following: coordinate i_ℓ of $\vec{x}$ matches the parity of $S|_{\mathrm{bin}\ i_\ell}$ for $1 \le \ell \le 3$, and coordinate i_4 of $\vec{x}$ is v. Let j satisfy $\vec{x} \in PS_j(4,2)$. We know such a j exists because $PS_0(4,2)$, $PS_1(4,2)$, $PS_2(4,2)$, $PS_3(4,2)$ partition the set of edges of Q_4. We claim that $S \in BinPS_j^{R(s)}(4,2)$ for some $0 \le s \le 4^d-1$. We know that $BinSetsPS_j^{R(s)}(4,2)$ contains a bin-form A_s which corresponds to $\vec{x}$ for each $0 \le s \le 4^d-1$. For example, if $S = (0,v,1,1,0,1,1,1,0,1,0,0,1,1,1,1)$, then $S|_{\mathrm{bin}\ 1} = (0,v,1,1) \in E(Q_4)$, $S|_{\mathrm{bin}\ 2} = (0,1,1,1) \in \mathbf{1}$, $S|_{\mathrm{bin}\ 3} = (0,1,0,0) \in \mathbf{1}$, and $S|_{\mathrm{bin}\ 4} = (1,1,1,1) \in \mathbf{0}$. Thus, $\vec{x} = (v,1,1,0) \in PS_3(4,2)$ and $BinSetsPS_3^{R(s)}(4,2)$ contains $A_s = (\mathbf{v}^{(R(s))}, \mathbf{1}, \mathbf{1}, \mathbf{0})$.

We have to prove that there exists a value of $0 \le s \le 4^d-1$ such that $S \in A_s$. Suppose that $S|_{\mathrm{bin}\ i_4} \in \mathbf{v}^{(k)}$. We know that such a k exists because $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(4^d-1)}$ partition $E(Q_{4^d})$. Let $s \in \{0, \dots, 4^d-1\}$ satisfy

$$(s + \mathrm{Index}(S|_{\mathrm{bin}\ i_1}) + \mathrm{Index}(S|_{\mathrm{bin}\ i_2}) + \mathrm{Index}(S|_{\mathrm{bin}\ i_3})) \bmod 4^d = k.$$

Thus, A_s contains the set of edges B whose entry in bin i_ℓ equals $S|_{\mathrm{bin}\ i_\ell}$ for $1 \le \ell \le 3$ and whose entry in bin i_4 is an element of $\mathbf{v}^{(k)}$. Since $S \in B$, $S \in A_s$. Therefore, $E(Q_{4^{d+1}}) \subseteq \bigcup_{j,s} BinPS_j^{R(s)}(4,2)$, and $|E(Q_{4^{d+1}})| = \left|\bigcup_{j,s} BinPS_j^{R(s)}(4,2)\right|$.

Since each $\mathbf{v}^{(j)}$ has the same cardinality and $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(4^d-1)}$ partition $E(Q_{4^d})$, which has cardinality $4^d(2^{4^d-1})$, $|\mathbf{v}^{(j)}| = 2^{4^d-1}$ for $0 \le j \le 4^d-1$. Each $BinSetsPS_j^{R(s)}(4,2)$ contains eight bin-forms. Since $|\mathbf{0}| = |\mathbf{1}| = 2^{4^d-1}$, each bin-form has cardinality $\left(2^{4^d-1}\right)^3 (2^{4^d-1})$.

Thus, $\left|BinPS_j^{R(s)}(4,2)\right| \le 8\left(2^{4^d-1}\right)^4 = 2^{4^{d+1}-1}$, and

$$\left|\bigcup_{j,s} BinPS_j^{R(s)}(4,2)\right| \le \sum_{j,s}\left|BinPS_j^{R(s)}(4,2)\right| \le 4^{d+1}2^{4^{d+1}-1} = \left|E(Q_{4^{d+1}})\right|.$$

Therefore, it must be the case that $\left|BinPS_j^{R(s)}(4,2)\right| = 2^{4^{d+1}-1}$ for each $0 \le j \le 3$ and $0 \le s \le 4^d - 1$, and the Breaker's win pairing strategies $BinPS_j^{R(s)}(4,2)$ form a partition of $E(Q_{4^{d+1}})$. $\square$

Lemma 2 *For all $k \ge 1$, the sets $\mathbf{0}_{2^k}$ and $\mathbf{1}_{2^k}$ can be partitioned into subsets $A_1, \ldots, A_{2^k}$ and $B_1, \ldots, B_{2^k}$, respectively, so that every subcube S of Q_{2^k} of dimension $2^{k-1}+1$ satisfies $V(S) \cap A_j \ne \emptyset$ and $V(S) \cap B_j \ne \emptyset$ for each $j \in [2^k]$, where $V(S)$ is the set of vertices of S.*

Proof of Lemma 2: We proceed by induction on k. The case $k = 1$ is trivial. Let $k \ge 2$. By the inductive hypothesis, there exists a partition of $\mathbf{0}_{2^{k-1}}$ into $A_1, \ldots, A_{2^{k-1}}$ and a partition of $\mathbf{1}_{2^{k-1}}$ into $B_1, \ldots, B_{2^{k-1}}$ so that every subcube S of $Q_{2^{k-1}}$ of dimension $2^{k-2}+1$ satisfies $V(S) \cap A_j \ne \emptyset$ and $V(S) \cap B_j \ne \emptyset$ for each $j \in [2^{k-1}]$.

For $X, Y \subseteq \{0,1\}^n$, let $(X,Y) = \{(\vec{x}, \vec{y}) : \vec{x} \in X, \vec{y} \in Y\}$, where $(\vec{x}, \vec{y})$ represents the $2n$-dimensional vector whose first n coordinates equal the coordinates of $\vec{x}$ and last n coordinates equal the coordinates of $\vec{y}$. For each $\ell \in [2^{k-1}]$, let

$$G_\ell = \bigcup_{(i,j)\in X_\ell} (A_i, A_j),$$

let

$$H_\ell = \bigcup_{(i,j)\in X_\ell} (B_i, B_j),$$

let

$$I_\ell = \bigcup_{(i,j)\in X_\ell} (A_i, B_j),$$

let

$$J_\ell = \bigcup_{(i,j)\in X_\ell} (B_i, A_j),$$

where

$$X_\ell = \{(i,j) \in [2^{k-1}]^2 : i + j \equiv \ell \pmod{2^{k-1}}\}.$$

We claim that $\{G_1, \ldots, G_{2^{k-1}}, H_1, \ldots, H_{2^{k-1}}\}$ is the desired partition of $\mathbf{0}_{2^k}$ and $\{I_1, \ldots, I_{2^{k-1}}, J_1, \ldots, J_{2^{k-1}}\}$ is the desired partition of $\mathbf{1}_{2^k}$.

We will only prove that $\{G_1, \ldots, G_{2^{k-1}}, H_1, \ldots, H_{2^{k-1}}\}$ is the desired partition of $\mathbf{0}_{2^k}$. Proving the result for $\mathbf{1}_{2^k}$ is similar. It is straightforward to check that $\{G_1, \ldots, G_{2^{k-1}}, H_1, \ldots, H_{2^{k-1}}\}$ is a partition of $\mathbf{0}_{2^k}$. For example, if $i \ne j$, then $G_i \cap G_j = \emptyset$ because if $\vec{x} \in G_i$ and $\vec{y} \in G_j$, then $\vec{x} \in (A_{i_1}, A_{i_2})$ where $i_1 + i_2 \equiv i \pmod{2^{k-1}}$ and $\vec{y} \in (A_{j_1}, A_{j_2})$ where $j_1 + j_2 \equiv j \pmod{2^{k-1}}$. Since $i \ne j$ and $\{i,j\} \subseteq [2^{k-1}]$, $i_1 + i_2 \not\equiv j_1 + j_2 \pmod{2^{k-1}}$. Thus, $i_1 \ne j_1$ or $i_2 \ne j_2$. W.l.o.g., $i_2 \ne j_2$, in which case $A_{i_2} \cap A_{j_2} = \emptyset$ and $\vec{x}$ does not equal

$\vec{y}$ because they are not equal in their last 2^{k-1} coordinates. We can use similar arguments to show that the sets in $\{G_1, \ldots, G_{2^{k-1}}, H_1, \ldots, H_{2^{k-1}}\}$ are pairwise disjoint.

To show that $G_1 \cup \cdots \cup G_{2^{k-1}} \cup H_1 \cup \cdots \cup H_{2^{k-1}} = \mathbf{0}_{2^k}$, suppose that $\vec{x} \in \mathbf{0}_{2^k}$. Let $\vec{x}_1$ be $\vec{x}$ restricted to its first 2^{k-1} coordinates, and let $\vec{x}_2$ be $\vec{x}$ restricted to its last 2^{k-1} coordinates, so that we can write $\vec{x} = (\vec{x}_1, \vec{x}_2)$ with a slight abuse of notation. Since $\vec{x} \in \mathbf{0}_{2^k}$, either $\{\vec{x}_1, \vec{x}_2\} \subseteq \mathbf{0}_{2^{k-1}}$ or $\{\vec{x}_1, \vec{x}_2\} \subseteq \mathbf{1}_{2^{k-1}}$. Since $A_1, \ldots, A_{2^{k-1}}$ is a partition of $\mathbf{0}_{2^{k-1}}$ and $B_1, \ldots, B_{2^{k-1}}$ is a partition of $\mathbf{1}_{2^{k-1}}$, either $\vec{x} \in (A_i, A_j)$ or $\vec{x} \in (B_i, B_j)$ for some $i, j \in [2^{k-1}]$.

Let S be a subcube of Q_{2^k} of dimension $2^{k-1}+1$. Similar to what we did in our previous proofs, divide the coordinates of S into two bins, so that bin 1 contains coordinates $1, \ldots, 2^{k-1}$ and bin 2 contains coordinates $2^{k-1}+1, \ldots, 2^k$. Since S has dimension $2^{k-1}+1$, there are $(2^{k-1}+1)$ $*$'s distributed between bin 1 and bin 2. Thus, $S|_{\text{bin 1}}$ and $S|_{\text{bin 2}}$ both have dimension at least 1. Additionally, exactly one of $S|_{\text{bin 1}}$ and $S|_{\text{bin 2}}$ has dimension at least $2^{k-2}+1$. W.l.o.g., $S|_{\text{bin 1}}$ has dimension at least $2^{k-2}+1$. Thus, $V(S|_{\text{bin 1}}) \cap A_i \neq \emptyset$ and $V(S|_{\text{bin 1}}) \cap B_i \neq \emptyset$ for each $i \in [2^{k-1}]$. Since $S|_{\text{bin 2}}$ has dimension at least 1, $V(S|_{\text{bin 2}}) \cap \mathbf{0}_{2^{k-1}} \neq \emptyset$ and $V(S|_{\text{bin 2}}) \cap \mathbf{1}_{2^{k-1}} \neq \emptyset$. Thus, $V(S|_{\text{bin 2}}) \cap A_{j_0} \neq \emptyset$ for some $j_0 \in [2^{k-1}]$ and $V(S|_{\text{bin 2}}) \cap B_{j_1} \neq \emptyset$ for some $j_1 \in [2^{k-1}]$. Therefore, $V(S) \cap G_\ell \neq \emptyset$ and $V(S) \cap H_\ell \neq \emptyset$ for each $\ell \in [2^{k-1}]$. □

5 Pairing Strategies for Specific Values of n and k

Both Theorems 1 and 2 require the existence of a Breaker's win pairing strategy for a game played on the vertices of Q_n to construct a Breaker's win pairing strategy for a game played on the vertices of Q_{4n}. The following two lemmas allow us to construct Breaker's win pairing strategies for games played on the vertices of Q_d where d is not divisible by 4.

Lemma 3 ([23]) *If there is a Breaker's win pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,k)$, then there is a Breaker's win pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n+1,k+1)$.*

Lemma 4 *If there exists a matching which is a Breaker's win pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(N,k)$, then there is a matching which is a Breaker's win pairing strategy for the Maker–Breaker game played on $\mathcal{Q}(n,k)$ for all $n \leq N$.*

Both lemmas are fairly easy to justify. For a full proof of Lemma 3, see [23]. To understand the idea behind Lemma 4, for example, observe that there is a natural correspondence between the set of k-dimensional subcubes of Q_n and the set of k-dimensional subcubes of Q_N whose last $N-n$ coordinates are fixed at 0. The set of edges from our Breaker's win pairing strategy which handles those k-dimensional subcubes must also have their last $N-n$ coordinates fixed at 0. If we truncate each of those edges after their n^{th} coordinate, we will obtain a Breaker's win pairing strategy for the set of k-dimensional subcubes of Q_n.

So far we have exhibited Breaker's win pairing strategies for $\mathcal{Q}(3,2)$, $\mathcal{Q}(4,2)$ and $\mathcal{Q}(9,4)$. Let us provide a Breaker's win pairing strategy for $\mathcal{Q}(6,3)$ in order to help us construct Breaker's win pairing strategies for other values of n and k.

To construct a Breaker's win pairing strategy for $\mathcal{Q}(6,3)$, we will use sets of edges that *resemble* cyclic permutations. For example, let

$$\begin{aligned}\langle(v,0,1,0,0,0)\rangle = \{&(v,0,1,0,0,0),\\ &(0,v,0,1,0,0),\\ &(0,0,v,0,1,0),\\ &(0,0,0,v,0,1),\\ &(1,0,0,0,v,0),\\ &(0,1,0,0,0,v)\}.\end{aligned}$$

Then,

$$\langle(v,0,1,0,0,0)\rangle \cup \langle(v,1,0,1,1,1)\rangle \cup \langle(v,0,1,1,0,0)\rangle \cup \langle(v,1,0,0,1,1)\rangle \tag{2}$$

is a Breaker's win pairing strategy for $\mathcal{Q}(6,3)$ consisting of 24 edges (verified by computer).

If we start with our Breaker's win pairing strategy for $\mathcal{Q}(6,3)$ and repeated apply Corollary 1(b), then we obtain a Breaker's win pairing strategy for $\mathcal{Q}(6\cdot 4^n, 2\cdot 4^n+1)$ for all $n \geq 0$. Likewise, if we start with our Breaker's win pairing strategy for $\mathcal{Q}(9,4)$ and repeated apply Corollary 1(b), then we obtain a Breaker's win pairing strategy for $\mathcal{Q}(9\cdot 4^n, 3\cdot 4^n+1)$ for all $n \geq 0$. Theorem 3 states that there is a Breaker's win pairing strategy for $\mathcal{Q}(4^{n+1}, 4^n+1)$ for all $n \geq 0$.

For each $n \geq 0$, there remain three intervals for which we have not yet described a Breaker's win pairing strategy, namely, for games played on the vertices of Q_d where $4^{n+1} < d < 6\cdot 4^n$, or $6\cdot 4^n < d < 9\cdot 4^n$, or $9\cdot 4^n < d < 4^{n+2}$. To establish the existence of Breaker's win pairing strategies for these values of d, we can use Lemmas 3 and 4. We use the same approach for each interval. Specifically, for an interval of the form $N_1 < d < N_2$, we have a Breaker's win pairing strategy for $\mathcal{Q}(N_1, k_1)$ and $\mathcal{Q}(N_2, k_2)$. When $d = N_1 + j$ for $1 \leq j \leq 4^n$, we use Lemma 3 and our Breaker's win pairing strategy for $\mathcal{Q}(N_1, k_1)$ to obtain a Breaker's win pairing strategy for $\mathcal{Q}(N_1 + j, k_1 + j)$. When $d = N_1 + j$ for $4^n + 1 \leq j < N_2$, we use Lemma 4 and our Breaker's win pairing strategy for $\mathcal{Q}(N_2, k_2)$ to obtain a Breaker's win pairing strategy for $\mathcal{Q}(N_1 + j, k_2)$.

After applying this technique to all three intervals, we obtain Breaker's win pairing strategies for

$$\begin{aligned}
&\mathcal{Q}(4^{n+1}+j, 4^n+1+j), && 1 \leq j \leq 4^n,\\
&\mathcal{Q}(4^{n+1}+j, 2\cdot 4^n+1), && 4^n+1 \leq j \leq 2\cdot 4^n,\\
&\mathcal{Q}(6\cdot 4^n+j, 2\cdot 4^n+1+j), && 1 \leq j \leq 4^n,\\
&\mathcal{Q}(6\cdot 4^n+j, 3\cdot 4^n+1), && 4^n+1 \leq j \leq 3\cdot 4^n,\\
&\mathcal{Q}(9\cdot 4^n+j, 3\cdot 4^n+1+j), && 1 \leq j \leq 4^n,\\
&\mathcal{Q}(9\cdot 4^n+j, 4^{n+1}+1), && 4^n+1 \leq j \leq 7\cdot 4^n.
\end{aligned}$$

Using the results stated in this section, we have established the existence of a non-trivial Breaker's win pairing strategy for $\mathcal{Q}(N,K)$ for each $N \geq 3$. When $N = 4^{n+1}$, we have that K is $N/4+1$. When $N = 6\cdot 4^n$ or $N = 9\cdot 4^n$, we have that K is $N/3+1$. We can ask the following question. What is the largest value that the ratio K/N attains using the results above? We observe that as N increases from 4^{n+1} to $5 \cdot 4^n$, the ratio K/N increases from $\frac{1}{4} + \frac{1}{N}$ to $\frac{2}{5} + \frac{1}{N}$. As N increases from $5 \cdot 4^n$ to $6 \cdot 4^n$, K/N decreases from $\frac{2}{5} + \frac{1}{N}$ to $\frac{1}{3} + \frac{1}{N}$. As N increases from $6 \cdot 4^n$ to $7 \cdot 4^n$, K/N increases from $\frac{1}{3} + \frac{1}{N}$ to $\frac{3}{7} + \frac{1}{N}$. As N increases from $7 \cdot 4^n$ to $9 \cdot 4^n$, K/N decreases from $\frac{3}{7} + \frac{1}{N}$ to $\frac{1}{3} + \frac{1}{N}$. As N increases from $9 \cdot 4^n$ to $10 \cdot 4^n$, K/N increases from $\frac{1}{3} + \frac{1}{N}$ to $\frac{2}{5} + \frac{1}{N}$. As N increases from $10 \cdot 4^n$ to 4^{n+2}, K/N decreases from $\frac{2}{5} + \frac{1}{N}$ to $\frac{1}{4} + \frac{1}{N}$. The largest value K/N achieves is $\frac{3}{7} + \frac{1}{N}$, when $N = 7 \cdot 4^n$ and $K = 3 \cdot 4^n + 1$. One can check that for each $N \geq 3$, there is a Breaker's win pairing strategy for $K = \left\lfloor \frac{3}{7} N \right\rfloor + 1$. We present the values of N and K corresponding to the (locally) minimum and (locally) maximum values achieved by K/N in the following table.

N	K
4^n	$N/4+1$
$6 \cdot 4^n$	$N/3+1$
$9 \cdot 4^n$	$N/3+1$
$5 \cdot 4^n$	$(2/5)N+1$
$10 \cdot 4^n$	$(2/5)N+1$
$7 \cdot 4^n$	$(3/7)N+1$

6 Extra Results

In Lemma 5, we state a generalization of Lemma 2. The proof of Lemma 5 is a fairly straightforward generalization of the proof of Lemma 2, but we include the proof for completeness.

Lemma 5 *For all $n \geq 1$ and all $c \geq 2$, the sets $\mathbf{0}_{c^n}$ and $\mathbf{1}_{c^n}$ can each be partitioned into $(2^{c-1})^n$ subsets so that every subcube S of Q_{c^n} of dimension $c^n - c^{n-1} + 1$ contains a vertex from each subset in each partition.*

Proof of Lemma 5: We proceed by induction on n. The case $n = 1$ is trivial. Let $n \geq 2$. By the inductive hypothesis, there exists a partition of $\mathbf{0}_{c^{n-1}}$ into $A_1, \ldots, A_{(2^{c-1})^{n-1}}$ and a partition of $\mathbf{1}_{c^{n-1}}$ into $B_1, \ldots, B_{(2^{c-1})^{n-1}}$ so that every subcube S of $Q_{c^{n-1}}$ of dimension $c^{n-1} - c^{n-2} + 1$ satisfies $V(S) \cap A_j \neq \emptyset$ and $V(S) \cap B_j \neq \emptyset$ for each $j \in [(2^{c-1})^{n-1}]$.

Suppose $X_i \subseteq \{0,1\}^N$ for $i \in [c]$. Let $(X_1, \ldots, X_c) = \{(\vec{x}_1, \ldots, \vec{x}_c) : \vec{x}_i \in X_i \text{ for each } i \in [c]\}$, where $(\vec{x}_1, \ldots, \vec{x}_c)$ represents the cN-dimensional vector whose coordinates in positions $(j-1)N + 1$ through jN equal the coordinates of $\vec{x}_j$ for each $j \in [c]$. For each vector $(b_1, \ldots, b_c) \in \{0,1\}^c$ and each vector of indices $(i_1, \ldots, i_c) \in [(2^{c-1})^{n-1}]^c$, define the set $(D_{i_1}, \ldots, D_{i_c})$ where $D_{i_j} = A_{i_j}$ if $b_j = 0$ and $D_{i_j} = B_{i_j}$ if $b_j = 1$. For example, if $c = 3$ and $n = 2$, we could have $(1,1,0) \in \{0,1\}^3$ and $(4,1,2) \in [4]^3$ which result in the set (B_4, B_1, A_2). Then for each $\vec{b} \in \mathbf{0}_c$ and each $\ell \in [(2^{c-1})^{n-1}]$, we define the set

$$A_{(\vec{b},\ell)} = \bigcup_{(i_1,\ldots,i_c) \in I_\ell} (D_{i_1}, \ldots, D_{i_c}),$$

where

$$I_\ell = \{(i_1, \ldots, i_c) \in [(2^{c-1})^{n-1}]^c : i_1 + \cdots + i_c \equiv \ell \pmod{(2^{c-1})^{n-1}}\}.$$

For example, in the case $c = 3$, we define 4^n sets

$$\begin{aligned}
A_{((0,0,0),\ell)} &= \bigcup_{(i_1,i_2,i_3)\in I_\ell} (A_{i_1}, A_{i_2}, A_{i_3}), && 1 \le \ell \le 4^{n-1},\\
A_{((0,1,1),\ell)} &= \bigcup_{(i_1,i_2,i_3)\in I_\ell} (A_{i_1}, B_{i_2}, B_{i_3}), && 1 \le \ell \le 4^{n-1},\\
A_{((1,0,1),\ell)} &= \bigcup_{(i_1,i_2,i_3)\in I_\ell} (B_{i_1}, A_{i_2}, B_{i_3}), && 1 \le \ell \le 4^{n-1},\\
A_{((1,1,0),\ell)} &= \bigcup_{(i_1,i_2,i_3)\in I_\ell} (B_{i_1}, B_{i_2}, A_{i_3}), && 1 \le \ell \le 4^{n-1},
\end{aligned}$$

where

$$I_\ell = \{(i_1, i_2, i_3) \in [4^{n-1}]^3 : i_1 + i_2 + i_3 \equiv \ell \pmod{4^{n-1}}\}.$$

For each $\vec{b} \in \mathbf{1}_c$ and each $\ell \in [(2^{c-1})^{n-1}]$, we define $(2^{c-1})^n$ sets $B_{(\vec{b},\ell)}$ similarly. We claim that the $(2^{c-1})^n$ sets $A_{(\vec{b},\ell)}$ partition $\mathbf{0}_{c^n}$ and the $(2^{c-1})^n$ sets $B_{(\vec{b},\ell)}$ partition $\mathbf{1}_{c^n}$. (We will only prove results for the sets $A_{(\vec{b},\ell)}$ since the corresponding proofs for the sets $B_{(\vec{b},\ell)}$ are similar.)

Suppose that $(\vec{b}_1, \ell_1) \neq (\vec{b}_2, \ell_2)$. We claim this implies $A_{(\vec{b}_1,\ell_1)} \cap A_{(\vec{b}_2,\ell_2)} = \emptyset$. If $\vec{b}_1 \neq \vec{b}_2$, it should be fairly obvious that $A_{(\vec{b}_1,\ell_1)} \cap A_{(\vec{b}_2,\ell_2)} = \emptyset$. Indeed, if $\vec{b}_1$ differs from $\vec{b}_2$ in coordinate k, then one of $A_{(\vec{b}_1,\ell_1)}$ and $A_{(\vec{b}_2,\ell_2)}$ will have A_{i_k} in coordinate k and the other will have $B_{i'_k}$ in coordinate k. Now suppose $\vec{b}_1 = \vec{b}_2$ but $\ell_1 \neq \ell_2$. In this case, $I_{\ell_1} \cap I_{\ell_2} = \emptyset$. Thus, for every $(D_{i_1}, \ldots, D_{i_c}) \subseteq A_{(\vec{b},\ell_1)}$ and every $(D_{i'_1}, \ldots, D_{i'_c}) \subseteq A_{(\vec{b},\ell_2)}$, there exists a coordinate k where $i_k \neq i'_k$. Thus, $D_{i_k} \cap D_{i'_k} = \emptyset$, because the A_i are pairwise disjoint and the B_i are pairwise disjoint, by the inductive hypothesis.

To show that

$$\bigcup_{(\vec{b},\ell)\in\mathbf{0}_c\times[(2^{c-1})^{n-1}]} A_{(\vec{b},\ell)} = \mathbf{0}_{c^n},$$

suppose that $\vec{x} \in \mathbf{0}_{c^n}$. As we have done before, let us partition the coordinates of $\vec{x}$ into c bins of size c^{n-1} and write $\vec{x} = (\vec{x}_1, \ldots, \vec{x}_c)$, where $\vec{x}_j$ is $\vec{x}$ restricted to bin j. Let b_j be the parity of $\vec{x}_j$ for each $j \in [c]$. Since $\vec{x} \in \mathbf{0}_{c^n}$, $(b_1, \ldots, b_c) \in \mathbf{0}_c$. Since $\bigcup A_i = \mathbf{0}_{c^{n-1}}$ and $\bigcup B_i = \mathbf{1}_{c^{n-1}}$, there exists an $\ell \in [(2^{c-1})^{n-1}]$ such that $\vec{x} \in A_{((b_1,\ldots,b_c),\ell)}$.

Let S be a subcube of Q_{c^n} of dimension $c^n - c^{n-1} + 1$. Partition the coordinates of S into c bins each of size c^{n-1}. There are $(c^n - c^{n-1} + 1)$ $*$'s distributed amongst the c bins. Since $c^n - c^{n-1} + 1 = (c-1)c^{n-1} + 1$, $S|_{\text{bin } j}$ has dimension at least 1 for each $j \in [c]$. Since $c^n - c^{n-1} + 1 = c(c^{n-1} - c^{n-2}) + 1$, $S|_{\text{bin } j}$ has dimension at least $c^{n-1} - c^{n-2} + 1$ for some $j \in [c]$. W.l.o.g., $S|_{\text{bin } c}$ has dimension at least $c^{n-1} - c^{n-2} + 1$. Thus, $V(S|_{\text{bin } c}) \cap A_i \neq \emptyset$ and $V(S|_{\text{bin } c}) \cap B_i \neq \emptyset$ for each $i \in [(2^{c-1})^{n-1}]$. For each $j \in [c-1]$, since $S|_{\text{bin } j}$ has dimension at least 1, $V(S|_{\text{bin } j}) \cap \mathbf{0}_{c^{n-1}} \neq \emptyset$ and $V(S|_{\text{bin } j}) \cap \mathbf{1}_{c^{n-1}} \neq \emptyset$. Thus, for each $j \in [c-1]$, $V(S|_{\text{bin } j}) \cap A_{i_j} \neq \emptyset$ for some $i_j \in [(2^{c-1})^{n-1}]$ and $V(S|_{\text{bin } j}) \cap B_{i'_j} \neq \emptyset$ for some $i'_j \in [(2^{c-1})^{n-1}]$.

Let $(b_1, \ldots, b_c) \in \mathbf{0}_c$ and $\ell \in [(2^{c-1})^{n-1}]$ both be arbitrary. W.l.o.g., suppose that $b_c = 1$. Then, for each $i \in [(2^{c-1})^{n-1}]$, let $(D_1, \ldots, D_{c-1}, B_i)$ satisfy $D_j = A_{i_j}$ if $b_j = 0$ and $D_j = B_{i'_j}$

if $b_j = 1$ for each $j \in [c-1]$, i.e., we choose D_j to be the appropriate set A_{i_j} or $B_{i'_j}$ from above which intersects $V(S|_{\text{bin } j})$. Observe that $V(S) \cap (D_1, \ldots, D_{c-1}, B_i) \neq \emptyset$ for each $i \in [(2^{c-1})^{n-1}]$, since $V(S|_{\text{bin } c})$ intersects every set B_i. Let $E = \{j \in [c-1] : b_j = 0\}$ and let $F = \{j \in [c-1] : b_j = 1\}$. Let $i \in [(2^{c-1})^{n-1}]$ satisfy $i + \sum_{j \in E} i_j + \sum_{j \in F} i'_j \equiv \ell$ (mod $(2^{c-1})^{n-1}$). For this value of i, $(D_1, \ldots, D_{c-1}, B_i) \subseteq A_{((b_1,\ldots,b_c),\ell)}$. Since $(b_1, \ldots, b_c)$ and ℓ were arbitrary, $V(S) \cap A_{(\vec{b},\ell)} \neq \emptyset$ for all $(\vec{b}, \ell) \in \mathbf{0}_c \times [(2^{c-1})^{n-1}]$. $\square$

In Theorem 4, we state a modified version of Theorem 3 for $Q_{3^{d+1}}$. The proof of Theorem 4 and its setup, is essentially the same as the proof of Theorem 3.

We begin by introducing the following Breaker's win pairing strategies:

$$PS_0(3,2) = \{(v,0,0), (1,v,1), (0,1,v)\},$$

$$PS_1(3,2) = \{(v,0,1), (0,v,0), (1,1,v)\},$$

$$PS_2(3,2) = \{(v,1,0), (0,v,1), (1,0,v)\},$$

$$PS_3(3,2) = \{(v,1,1), (1,v,0), (0,0,v)\}.$$

In our example at the beginning of Section 4, we observed that these matchings partition the set of edges of Q_3, and each $PS_j(3,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(3,2)$ for $0 \leq j \leq 3$. We define the following sets of bin-forms based on $PS_j(3,2)$ for $0 \leq j \leq 3$. For example, when $j = 0$, let

$$BinSetsPS_0^{R(s)}(3,2) = \{(\mathbf{v}^{R(s)}, \mathbf{0}, \mathbf{0}), (\mathbf{1}, \mathbf{v}^{R(s)}, \mathbf{1}), (\mathbf{0}, \mathbf{1}, \mathbf{v}^{R(s)})\},$$

and

$$BinPS_0^{R(s)}(3,2) = (\mathbf{v}^{R(s)}, \mathbf{0}, \mathbf{0}) \cup (\mathbf{1}, \mathbf{v}^{R(s)}, \mathbf{1}) \cup (\mathbf{0}, \mathbf{1}, \mathbf{v}^{R(s)}),$$

where, for example,

$$\begin{aligned}(\mathbf{0}, \mathbf{1}, \mathbf{v}^{R(s)}) = \{(\vec{x}, \vec{y}, \vec{z}) : \vec{x} \in \mathbf{0}, \vec{y} \in \mathbf{1}, \vec{z} \in \mathbf{v}^{(j)}, \\ \text{where } j = (s + \text{Index}(\vec{x}) + \text{Index}(\vec{y})) \bmod m\}.\end{aligned}$$

These definitions will assume that we have 4^d matchings $\mathbf{v}^{(j)}$ (of equal cardinality) which partition the edges of Q_{3^d} and each $\mathbf{v}^{(j)}$ is a Breaker's win pairing strategy for $\mathcal{Q}(3^d, 3^{d-1}+1)$ in order to produce 4^{d+1} Breaker's win pairing strategies for $\mathcal{Q}(3^{d+1}, 3^d + 1)$.

Theorem 4 *For each $d \geq 0$, there exist 4^{d+1} disjoint Breaker's win pairing strategies for $\mathcal{Q}(3^{d+1}, 3^d + 1)$ with equal cardinalities which partition the set of edges of $Q_{3^{d+1}}$.*

Proof of Theorem 4: We proceed by induction on d. The Breaker's win pairing strategies $PS_j(3,2)$ for $0 \leq j \leq 3$ handle the case $d = 0$. Let $d \geq 1$. By the inductive hypothesis, there exist 4^d disjoint Breaker's win pairing strategies $\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(4^d-1)}$ for $\mathcal{Q}(3^d, 3^{d-1}+1)$ with equal cardinalities which partition the set of edges of Q_{3^d}. We will show that $BinPS_j^{R(s)}(3,2)$ is a Breaker's win pairing strategy for $\mathcal{Q}(3^{d+1}, 3^d + 1)$ for $0 \leq j \leq 3$ and $0 \leq s \leq 4^d - 1$, where we use $\mathbf{0}_{3^d}$, $\mathbf{1}_{3^d}$, and the Breaker's win pairing strategies $\mathbf{v}^{(j)}$ from the inductive hypothesis in the definitions of the bin-forms. Moreover, we will show that the Breaker's win pairing strategies $BinPS_j^{R(s)}(3,2)$ form a partition of the edges of $Q_{3^{d+1}}$.

Let S be a (3^d+1)-dimensional subcube of $Q_{3^{d+1}}$. We partition the coordinates of S into three bins each of size 3^d, so that bin j contains coordinates $1+(j-1)3^d$ through $j3^d$ for $j \in [3]$. Suppose $S|_{\text{bin } j}$ has f fixed coordinates. Bin j is full if $f = 3^d$, heavy if $3^d - 3^{d-1} \leq f \leq 3^d - 1$, and light if $0 \leq f \leq 3^d - 3^{d-1} - 1$. The subcube S has $3^{d+1} - 3^d - 1$ fixed coordinates. Since $3^{d+1} - 3^d - 1 = 3(3^d - 3^{d-1}) - 1$, at least one bin is light. Since $3^{d+1} - 3^d - 1 = 2 \cdot 3^d - 1$, at most one bin is full. Thus, there are three cases to consider: (F, L, L), (H, H, L), and (F, H, L).

Suppose bin i_1 is full, and bins i_2 and i_3 are light. Let $A \in BinSetsPS_j^{R(s)}(3,2)$ be a bin-form which contains a bin-set in bin i_1 that matches the parity of $S|_{\text{bin } i_1}$. Since the other two bins are light, they can be handled by any bin-set. Thus, A handles S.

Suppose bins i_1 and i_2 are heavy, and bin i_3 is light. Let $A \in BinSetsPS_j^{R(s)}(3,2)$ contain $\mathbf{v}^{R(s)}$ in bin i_3. Since bin i_3 of S is light, $\mathbf{v}^{R(s)}$ can handle $S|_{\text{bin } i_3}$. In each of the other bins, A has either a $\mathbf{0}$ or a $\mathbf{1}$. Since each of $\mathbf{0}$ and $\mathbf{1}$ can handle any non-full-bin, A handles S.

Suppose bin i_1 is full, bin i_2 is heavy, and bin i_3 is light. Let $A \in BinSetsPS_j^{R(s)}(3,2)$ contain a bin-set in bin i_1 that matches the parity of $S|_{\text{bin } i_1}$. If A contains $\mathbf{0}$ or $\mathbf{1}$ in bin i_2, then A handles S. So, we may assume A contains $\mathbf{v}^{R(s)}$ in bin i_2. Since $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(4^d-1)}$ partition the set of edges of Q_{3^d} and $S|_{\text{bin } i_2}$ has dimension at least 1, there is a bin-set $\mathbf{v}^{(k)}$ which contains an edge that can handle $S|_{\text{bin } i_2}$. Let $c \in \{0, \dots, 4^d - 1\}$ satisfy $(s + \text{Index}(S|_{\text{bin } i_1}) + c) \bmod 4^d = k$. W.l.o.g., suppose A contains $\mathbf{0}$ in bin i_3. By Lemma 5, there is a partition of $\mathbf{0}_{3^d}$ into 4^d subsets such that every subcube of Q_{3^d} of dimension $3^d - 3^{d-1} + 1$ contains at least one vertex from each of the sets in the partition. Since bin i_1 is full and bin i_2 is heavy, there are at most $3^{d-1} - 1$ fixed coordinates in bin i_3, i.e., $S|_{\text{bin } i_3}$ has dimension at least $3^d - 3^{d-1} + 1$. Thus, there is a vertex $\vec{x} \in \mathbf{0}^{(c)}$ which handles $S|_{\text{bin } i_3}$. Therefore, we can find an edge in A that handles S.

Let $E(Q_{3^{d+1}})$ be the set of edges of $Q_{3^{d+1}}$. We will show that

$$E(Q_{3^{d+1}}) \subseteq \bigcup_{j,s} BinPS_j^{R(s)}(3,2),$$

which implies $\bigcup_{j,s} BinPS_j^{R(s)}(3,2) = E(Q_{3^{d+1}})$.

Let $S \in E(Q_{3^{d+1}})$. Suppose that $S|_{\text{bin } i_1}$ and $S|_{\text{bin } i_2}$ are both vertices in Q_{3^d} and $S|_{\text{bin } i_3}$ is an edge in Q_{3^d}. Let $\vec{x}$ be the edge in Q_3 which satisfies the following: coordinate i_ℓ of $\vec{x}$ matches the parity of $S|_{\text{bin } i_\ell}$ for $\ell \in \{1, 2\}$, and coordinate i_3 of $\vec{x}$ is v. Let j satisfy $\vec{x} \in PS_j(3,2)$. We know such a j exists because $PS_0(3,2)$, $PS_1(3,2)$, $PS_2(3,2)$, $PS_3(3,2)$ partition the set of edges of Q_3. We claim that $S \in BinPS_j^{R(s)}(3,2)$ for some $0 \leq s \leq 4^d - 1$. We know that $BinSetsPS_j^{R(s)}(3,2)$ contains a bin-form A_s which corresponds to $\vec{x}$ for each $0 \leq s \leq 4^d - 1$.

We have to prove that there exists a value of $0 \leq s \leq 4^d - 1$ such that $S \in A_s$. Suppose that $S|_{\text{bin } i_3} \in \mathbf{v}^{(k)}$. We know that such a k exists because $\mathbf{v}^{(0)}, \dots, \mathbf{v}^{(4^d-1)}$ partition $E(Q_{3^d})$. Let $s \in \{0, \dots, 4^d - 1\}$ satisfy

$$(s + \text{Index}(S|_{\text{bin } i_1}) + \text{Index}(S|_{\text{bin } i_2})) \bmod 4^d = k.$$

Thus, A_s contains the set of edges B whose entry in bin i_ℓ equals $S|_{\text{bin } i_\ell}$ for $\ell \in \{1,2\}$ and whose entry in bin i_3 is an element of $\mathbf{v}^{(k)}$. Since $S \in B$, $S \in A_s$. Therefore, $E(Q_{3^{d+1}}) \subseteq \bigcup_{j,s} BinPS_j^{R(s)}(3,2)$, and $|E(Q_{3^{d+1}})| = \left|\bigcup_{j,s} BinPS_j^{R(s)}(3,2)\right|$.

Since each $\mathbf{v}^{(i)}$ has the same cardinality and $\mathbf{v}^{(0)}, \ldots, \mathbf{v}^{(4^d-1)}$ partition $E(Q_{3^d})$, which has cardinality $3^d(2^{3^d-1})$, $|\mathbf{v}^{(i)}| = 3^d 2^{(3^d-2d-1)}$ for $0 \le i \le 4^d - 1$. Each $BinSetsPS_j^{R(s)}(3,2)$ contains three bin-forms. Since $|\mathbf{0}| = |\mathbf{1}| = 2^{3^d-1}$, each bin-form has cardinality $\left(2^{3^d-1}\right)^2 (3^d 2^{(3^d-2d-1)})$. Thus, $\left|BinPS_j^{R(s)}(4,2)\right| \le 3(2^{2\cdot 3^d-2})(3^d 2^{(3^d-2d-1)}) = 3^{d+1}2^{3^{d+1}-2(d+1)-1}$, and

$$\left|\bigcup_{j,s} BinPS_j^{R(s)}(3,2)\right| \le \sum_{j,s} \left|BinPS_j^{R(s)}(3,2)\right| \le 3^{d+1}2^{3^{d+1}-1} = |E(Q_{3^{d+1}})|.$$

Therefore, it must be the case that $\left|BinPS_j^{R(s)}(3,2)\right| = 3^{d+1}2^{3^{d+1}-2(d+1)-1}$ for each $0 \le j \le 3$ and $0 \le s \le 4^d - 1$, and the Breaker's win pairing strategies $BinPS_j^{R(s)}(3,2)$ form a partition of $E(Q_{3^{d+1}})$. □

7 Conclusion

Let $ps(n)$ be the smallest value of k such that Breaker wins the positional game on $\mathcal{Q}(n,k)$ by *using a pairing strategy.* We have proven the following upper bounds. If $n \in \{4^{d+1} : d \in \mathbb{N}\}$, then $ps(n) \le \frac{n}{4} + 1$. If $n \in \{3^{d+1} : d \in \mathbb{N}\} \cup \{6 \cdot 4^d : d \in \mathbb{N}\} \cup \{9 \cdot 4^d : d \in \mathbb{N}\}$, then $ps(n) \le \frac{n}{3} + 1$. In general, for all $n \ge 3$, $ps(n) \le \frac{3}{7}n + 1$. To obtain a lower bound on $ps(n)$, we cite Proposition 9 in [14], which implies that $ps(n) > \ln(n)$. Thus, there is a large gap between the upper and lower bounds on $ps(n)$ for most values of n. It would be interesting to improve any of these bounds. With regards to small specific values of n, because Maker has a winning strategy for $\mathcal{Q}(5,2)$ (see [23]) and $\mathcal{Q}(2,1)$, we know that $ps(3) = ps(4) = 2$ and $ps(5) = ps(6) = 3$. It would be nice to also determine the exact values of, say, $ps(7)$ and $ps(8)$.

We note that there is no direct analogue to Theorems 3 and 4 for $\mathcal{Q}(c^{d+1}, c^d+1)$ for $c \ge 5$ using our proof method. Indeed, Theorems 3 and 4 rely on the Breaker's win pairing strategies for $\mathcal{Q}(4,2)$ and $\mathcal{Q}(3,2)$ in order to create $BinSetsPS_j^{R(s)}(4,2)$ and $BinSetsPS_j^{R(s)}(3,2)$. Since Maker has a winning strategy for $\mathcal{Q}(n,2)$ for all $n \ge 5$, there are no Breaker's win pairing strategies for $\mathcal{Q}(n,2)$ from which we would create the bin-forms for $BinSetsPS_j^{(s)}(n,2)$ for all $n \ge 5$.

As a final note, we mention that some of our results can be viewed as being related to a Turán-type problem on Q_n. Let $\text{ex}(G,H)$ be the maximum number of edges in a subgraph of G which does not contain a copy of H. In [15], Erdős discussed some problems that he believed deserved more attention, including determining $\text{ex}(Q_n, C_4)$, which he conjectured to be $(\frac{1}{2} + o(1))|E(Q_n)|$. Much work has been done related to determining $\text{ex}(Q_n, C_{2t})$, see for example, Alon, Radoiŏić, Sudakov, and Vondraák [2], Axenovich and Martin [3], Balogh, Hu, Lidický, and Liu [4], Bialostocki [7], Brass, Harborth, and Nienborg [8], Brouwer, Dejter, and Thomassen [9], Chung [11], Conder [12], Conlon [13], Füredi and Özkahya [17, 18], and Thomason and Wagner [27].

In [1], Alon, Krech, and Szabó change the focus to studying $\mathrm{ex}(Q_n, Q_d)$. In particular, let $c(n,d)$ be the minimum number of edges that must be deleted from Q_n so that no copy of Q_d remains, and let $c_d = \lim_{n\to\infty} c(n,d)/|E(Q_n)|$. (For a study of $c(n,d)$ in a computer science context, see Graham, Harary, Livingston, and Stout [20].) In their approach, Alon, Krech, and Szabó used a "Ramsey-type framework," which involved studying *d-polychromatic* colorings of the edges of Q_n, i.e., colorings in which every d-dimensional subcube of Q_n contains an edge from every color class. They define $pc(n,d)$ to be the largest integer p such that there exists a d-polychromatic coloring of the edges of Q_n in p colors, and $p_d = \lim_{n\to\infty} pc(n,d)$. They also define higher-dimensional analogues, where the definition of $pc^{(\ell)}(n,d)$ is based on coloring each ℓ-dimensional subcube of Q_n so that each d-dimensional subcube contains an ℓ-dimensional subcube of each color. Thus, $pc(n,d)$ is the special case $\ell = 1$. They proved upper and lower bounds for p_d for all $d \geq 1$ and that $p_d^{(0)} = d+1$ for all $d \geq 0$. In [24], Offner proved that p_d equals the lower bound given by Alon, Krech, and Szabó. Much work related to polychromatic colorings on the hypercube has been done, for example, Chen [10], Goldwasser, Lidický, Martin, Offner, Talbot, and Young [19], Han and Offner [21], Offner [25], and Özkahya and Stanton [26].

We note that Theorems 4 and 3 provide a (3^d+1)-polychromatic *proper* coloring of $Q_{3^{d+1}}$ and a (4^d+1)-polychromatic proper coloring of $Q_{4^{d+1}}$ for all $d \geq 0$, both using 4^{d+1} colors, i.e., each color class forms a matching. It would be interesting to determine for which values of n and d there exists a d-polychromatic *proper* coloring of Q_n.

We also note that Lemma 5 provides a $(c^n - c^{n-1}+1)$-polychromatic coloring of the *vertices* of Q_{c^n} using $(2^{c-1})^n$ colors and only vertices from $\mathbf{0}_{c^n}$ (or $\mathbf{1}_{c^n}$). If we let $A_1, \ldots, A_{(2^{c-1})^n}$ be the partition of $\mathbf{0}_{c^n}$ and $B_1, \ldots, B_{(2^{c-1})^n}$ be the partition of $\mathbf{1}_{c^n}$, then $A_1 \cup B_1, \ldots, A_{(2^{c-1})^n} \cup B_{(2^{c-1})^n}$ works as a sort of $(c^n - c^{n-1} + 1)$-polychromatic *double-coloring* of the vertices of Q_{c^n} using $(2^{c-1})^n$ colors, i.e., every $(c^n - c^{n-1} + 1)$-dimensional subcube contains *two* vertices from each color class. It could be interesting to ask for which values of n, d, and p do there exist d-polychromatic double-colorings of Q_n using p colors.

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