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My Three Favorite Calculus Problems

Three of my favorite Calculus problems engage three very different learning styles of the students while addressing particularly important topics: techniques of integration, limits and differentiation/anti-differentiation rules. My first question engages students' understanding of the concept of function and is aimed at those who feel comfortable with symbolic manipulation. Despite the presence of an unknown mystery function in the integrand, students realize that regardless of what this function is, they have enough information to still evaluate the given integrals using either integration by substitution or intergration by parts. In the second problem, students who engage verbally or linguistically with the material are given the opportunity to parse and comment on an exchange of ideas between two students on a particularly important topic: the nature of limits. This problem also gets to the heart of L'Hôpital's Rule and the definition of a continuous function. The last problem is for visual learners who discern patterns quickly. By placing a set of related differentiation/anti-differentiation problems together the students can hopefully discern the relatedness between the differentiation and anti-differentiation operations in a very specific way. The central theme of this problem is the Chain Rule. For good measure, the notion of parameter (as opposed to variable) is also a part of this problem as well as the concept of generalization of an observed pattern.

1. Given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u}\,du = 3, \quad \int_1^2 g(u)\,du = 4, \quad \int_1^4 g(u)\,du = 5, \quad g(1) = 2, \quad g(2) = -2,$$

(a) Evaluate $I = \int_1^2 \ln(x) g'(x)\,dx$.

(b) Evaluate $J = \int_1^2 x g(x^2)\,dx$.

2. Two students are feverishly studying for their Calculus Final and have a disagreement about limits. Below is an excerpt of their conversation:

Vaughn: **I don't understand why we have to be careful when we evaluate a limit. I think we can always just evaluate** $\lim_{x \to c} f(x)$ **by evaluating** $f(c)$**.**

Sydney: **Oh yeah? How about the function** $f(x) = x^x$**? Okay, smarty-pants, what's the value of** $\lim_{x \to 0^+} x^x$**? Using "Vaughn's Law" the answer would be** $f(0) = 0^0$ **which I guess is zero, since zero times anything is zero.**

Vaughn: **Hmmm, well, no, your limit is a special example (that's why you have that little plus sign above the zero!) of what the professor called an indeterminate limit so you have to use something called Newton's Rule to evaluate it. All you do is re-write** x^x **as** $e^{x \ln(x)}$ **AND THEN you plug in zero for** x **and get** $e^{0 \ln(0)} = e^0 = 1$ **so the value of your limit is 1, not 0.**

Sydney: **I don't think you are using the rule correctly for evaluating indeterminate limits. I'm pretty sure it involves comparing the rates at which two competing functions approach their limits and so it involves derivatives. I just wish there was someway we could use our calculators to estimate the limit accurately but my cheap calculator doesn't have a** `LIM` **button, so we have absolutely no way of figuring out which of us has the right value of this limit.**

Vaughn: **Well, limits don't have anything to do with derivatives anyway so maybe it won't be an important Calculus concept to know for the final exam. Let's study something much more important like how to pronounce "Euler"!**

Write **at least 5 sentences** discussing which student you think has a better understanding of Calculus. Identify **any** and all correct, incorrect or partly correct statements made by the students. If a statement is incorrect explain why. **You must be careful not to make any incorrect statements yourself in your explanation.** PROOFREAD YOUR ANSWER.

3. Complete the following table of derivatives and anti-derivatives. A, B, C and D are (known) constants. **You should use the space on the other side of this page to check your answers**

	$f'(x)$	$f(x)$	$\int f(x)\,dx$
0.	$-\frac{1}{x^2}$	$\frac{1}{x}$	$\ln(x)$
1.		$\frac{1}{2x}$	
2.		$\frac{1}{2x+3}$	
3.		$\frac{1}{5x+4}$	
4.		$\frac{1}{Ax+B}$	
5.	$-\frac{C^2}{(Cx+D)^2}$		
	$g''(x)$	$g'(x)$	$g(x)$

ANSWER SHEET

1. Given the following information about an unknown function $g(x)$:

$$\int_1^2 \frac{g(u)}{u}\,du = 3, \quad \int_1^2 g(u)\,du = 4, \quad \int_1^4 g(u)\,du = 5, \quad g(1) = 2, \quad g(2) = -2$$

(a) Evaluate $I = \int_1^2 \ln(x)g'(x)\,dx$.

The first choice to make is whether to use Integration by Parts or Integration By Substitution. Since the integrand is a product of two functions but you don't see any composite functions the correct choice is IBP.

Then you have to decide which function are you going to anti-differentiate and which will you differentiate.

Since $g'(x)$ is a derivative, it will be easy to anti-differentiate, and $\ln(x)$ will be easy to differentiate.

So, let $u' = g'(x)$ and $v = \ln(x)$ then the IBP formula is $\int_a^b u'(x)v(x)\,dx = u(x)v(x)|_a^b - \int_a^b u(x)v'(x)\,dx$. Given our choice for u' and v means that $u(x) = g(x)$ and $v'(x) = \frac{1}{x}$

$$\begin{aligned} I = \int_1^2 \ln(x)g'(x)dx &= g(x)\ln(x)|_1^2 - \int_1^2 g(x)\frac{1}{x}dx \\ &= g(2)\ln(2) - g(1)\ln(1) - 3 \\ I &= -2\ln(2) - 3 \end{aligned}$$

(b) Evaluate $J = \int_1^2 xg(x^2)\,dx$.

Again, the first choice to make is whether to use Integration by Parts or Integration By Substitution.

Since part (a) was IBP it is likely that part (b) uses IBS, which when you examine the integrand becomes even more likely to be evaluated using IBS when a composite function $g(x^2)$ is observed in the integrand.

Let $u = x^2$. Then $du = 2x\,dx$ so $\frac{du}{2x} = dx$ and when $x = 1, \quad u = 1^2 = 1$ and when $x = 2, \quad u = 2^2 = 4$.

The entire integral is transformed from x-variables into u-variables.

$$\begin{aligned} J = \int_1^2 xg(x^2)\,dx &= \int_1^4 xg(u)\,\frac{du}{2x} \\ &= \int_1^4 g(u)\,\frac{du}{2} \\ &= \frac{1}{2}\int_1^4 g(u)\,du \\ &= \frac{1}{2}5 \\ J &= \frac{5}{2} \end{aligned}$$

ANSWER SHEET

2. Two students are feverishly studying for their Calculus Final and have a disagreement about limits. Below is an excerpt of their conversation:

Vaughn: **I don't understand why we have to be careful when we evaluate a limit. I think we can always just evaluate $\lim_{x\to c} f(x)$ by evaluating $f(c)$.**

Sydney: **Oh yeah? How about the function $f(x) = x^x$? Okay, smarty-pants, what's the value of $\lim_{x\to 0+} x^x$? Using "Vaughn's Law" the answer would be $f(0) = 0^0$ which I guess is zero, since zero times anything is zero.**

Vaughn: **Hmmm, well, no, your limit is a special example (that's why you have that little plus sign above the zero!) of what the professor called an indeterminate limit so you have to use something called Newton's Rule to evaluate it. All you do is re-write x^x as $e^{x\ln(x)}$ AND THEN you plug in zero for x and get $e^{0\ln(0)} = e^0 = 1$ so the value of your limit is 1, not 0.**

Sydney: **I don't think you are using the rule correctly for evaluating indeterminate limits. I'm pretty sure it involves comparing the rates at which two competing functions approach their limits and so it involves derivatives. I just wish there was someway we could use our calculators to estimate the limit accurately but my cheap calculator doesn't have a `LIM` button, so we have absolutely no way of figuring out which of us has the right value of this limit.**

Vaughn: **Well, limits don't have anything to do with derivatives anyway so maybe it won't be an important Calculus concept to know for the final exam. Let's study something much more important like how to pronounce "Euler"!**

Write **at least 5 sentences** discussing which student you think has a better understanding of Calculus. Identify **any** and all correct, incorrect or partly correct statements made by the students. If a statement is incorrect explain why. **You must be careful not to make any incorrect statements yourself in your explanation.** PROOFREAD YOUR ANSWER.

As in other things, Sydney is superior to Vaughn. Vaughn's first statement is FALSE since it is not always true. If the function in question $f(x)$ is CONTINUOUS at $x = c$ then Vaughn's statement would be true, but he doesn't mention this (or any other) caveat on what functions his statement applies to. In fact, that is the definition of continuity: $\lim_{x\to c} f(x) = f(c)$. Sydney correctly provides a counter-example of a function for which Vaughn's first statement is false but although it is generally true that zero times any finite number is zero, this does NOT mean that 0^0 is 0, it is *indeterminate*. Vaughn doesn't understand that the + sign above the zero means that Sydney is correctly only considering a right-sided limit, since the function x^x is clearly only well-defined for values $x > 0$. Vaughn correctly (somehow!) remembers that this particular limit is called "indeterminate" but then mis-remembers the name of the Rule used to evaluate indeterminate limits. It is **L'Hôpital's Rule**, not Newton's Rule. Vaughn does use the correct technique of re-writing x^x as $e^{x\ln x}$ and actually obtains the correct value of the limit but by using inaccurate means. You can't just plug-in zero into $x\ln(x)$ because $\ln(0)$ is undefined! The correct way to evaluate $\lim_{x\to 0+} x\ln(x)$ is to re-write it as $\lim_{x\to 0+} x\ln(x) = \lim_{x\to 0+} \frac{\ln(x)}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0$. Sydney is correct that evaluating indeterminate limits involves the competing rates at which two functions approach their limit but is incorrect when she says that she can't use her calculator to estimate the limit accurately. She could make a table and plug in values for x getting closer and closer to 0 from the right into x^x and see what the trend shows her. Some values would be $0.1^{0.1}, 0.01^{0.01}, 0.001^{0.001}, \ldots$. It's very clear that these numbers approach the value 1. Vaughn is wrong again when he says that limits have nothing to do with derivatives, the very definition of the derivative involves limits, i.e. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ and limits **are** a very important concept for a Calculus final exam.

ANSWER SHEET

3. Complete the following table of derivatives and anti-derivatives. A, B, C and D are (known) constants. **You should use the space on the other side of this page to check your answers**

	$f'(x)$	$f(x)$	$\int f(x)\,dx$
0.	$-\frac{1}{x^2}$	$\frac{1}{x}$	$\ln(x)$
1.	$-\frac{1}{(2x)^2}\cdot 2 = -\frac{1}{2x^2}$	$\frac{1}{2x}$	$\frac{1}{2}\ln 2x$
2.	$-\frac{2}{(2x+3)^2}$	$\frac{1}{2x+3}$	$\frac{1}{2}\ln(2x+3)$
3.	$-\frac{5}{(5x+4)^2}$	$\frac{1}{5x+4}$	$\frac{1}{5}\ln(5x+4)$
4.	$-\frac{A}{(Ax+B)^2}$	$\frac{1}{Ax+B}$	$\frac{1}{A}\ln(Ax+B)$
5.	$-\frac{C^2}{(Cx+D)^2}$	$\frac{C}{Cx+D}$	$\ln(Cx+D)$
	$g''(x)$	$g'(x)$	$g(x)$