An Introduction and Analysis of the Fast Fourier Transform

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Outline

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4. Fast Fourier Transform
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6. Examples comparing real time complexity
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7. FFT application on Solar Array data
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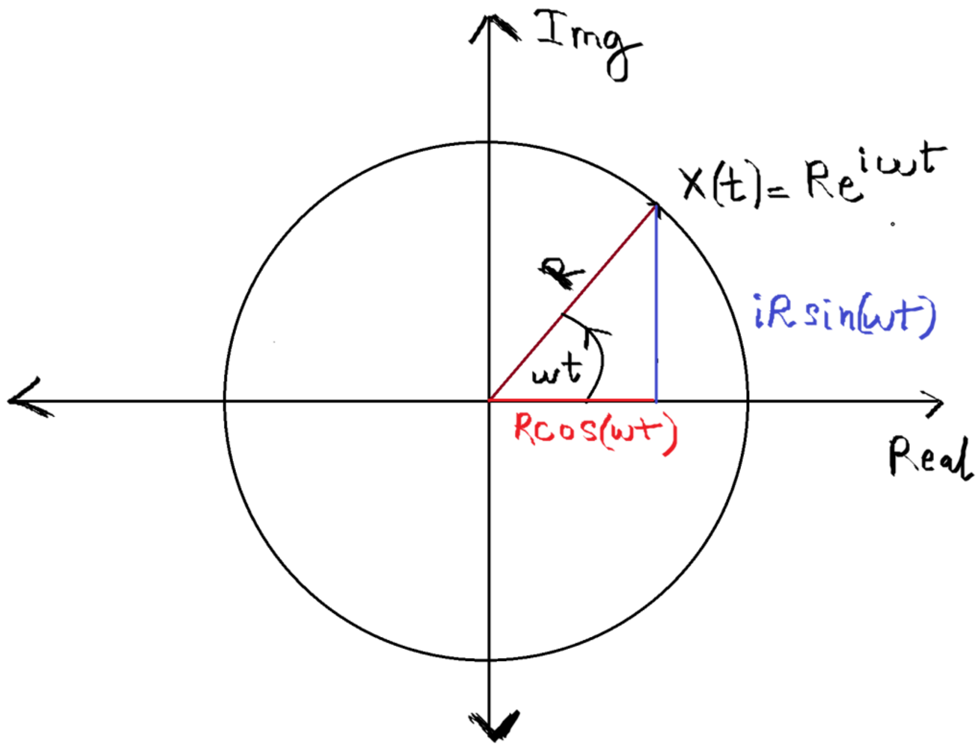
Continuous Fourier Transform



In the Polar coordinate with $r = R$ and $\theta = wt$

$$x(t) = Re^{iwt}$$

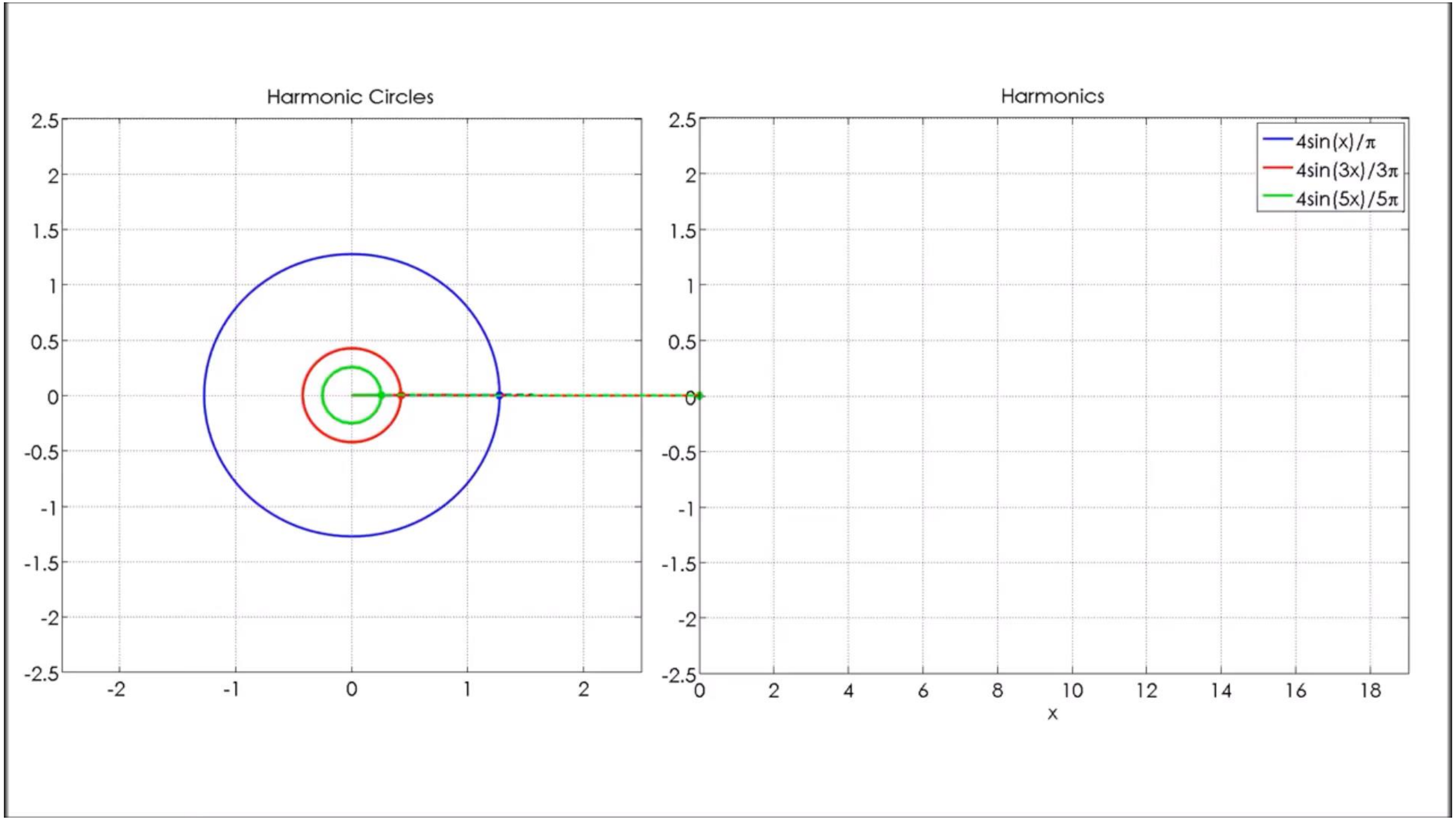
In the Cartesian coordinate

$$x(t) = Rcos(wt) + iRsin(wt)$$

Continuous Fourier Transform

The position *x* of time *t* is a periodic function of time.

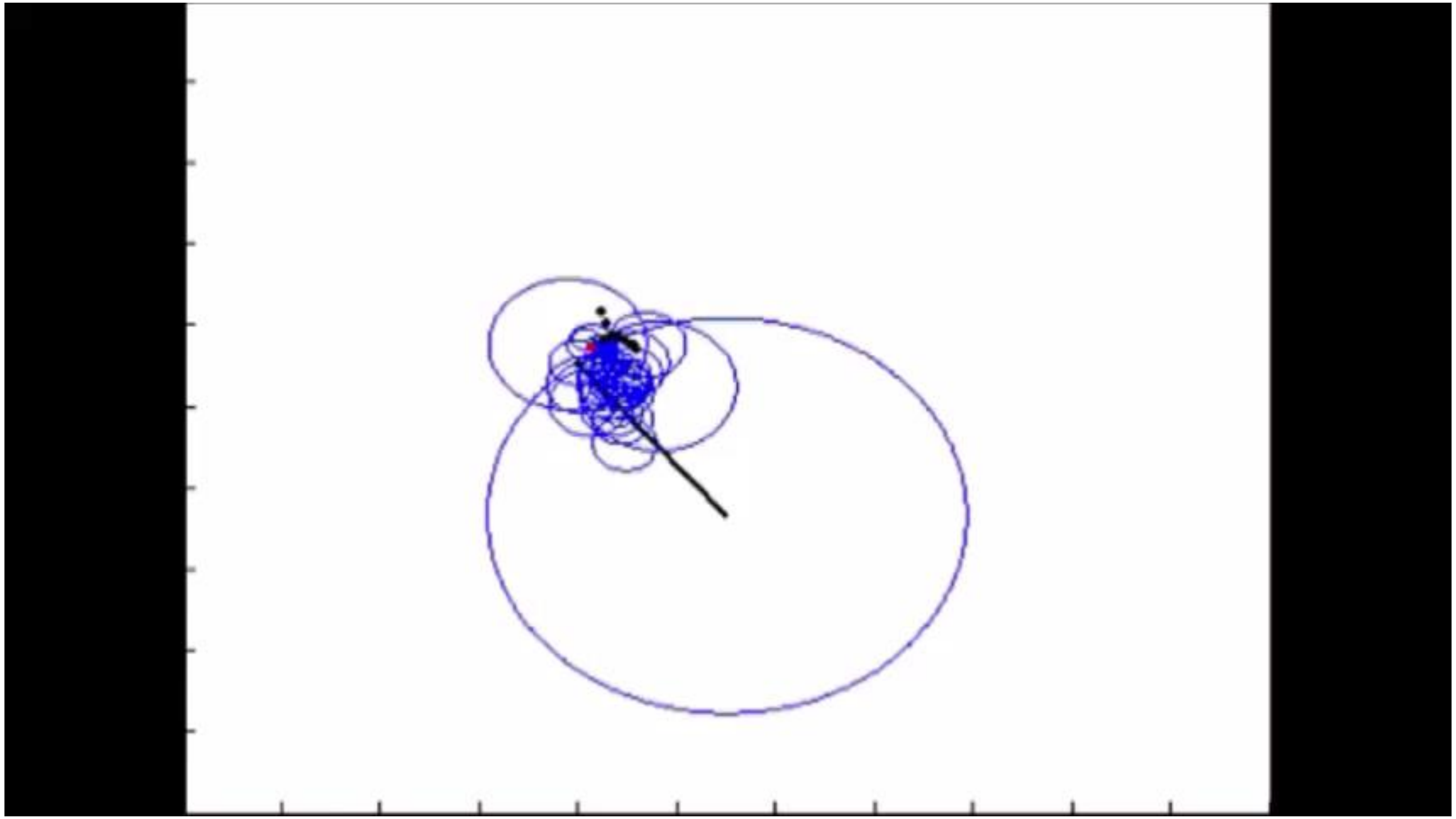
(https://youtu.be/LznjC4Lo7lE?t=28s)



Continuous Fourier Transform

If we add enough circles with different sizes and given appropriate angular frequency for each circle, we can create any shape [5].

(https://www.youtube.com/watch?v=QVuU2YCwHjw)



Continuous Fourier Transform

The path *x(t)* has *n* different circles each with unique angular frequency w_j, and radius R_j, can be described on the complex plane as a sum of all the circles:

$$x(t) = \sum_{j=1}^{n} R_j e^{i w_j t} \quad (1)$$

For a non-periodic function *x(t)*, we can assume that its period is approaching infinity, so it is represented by an infinite number of circles.

$$x(t) = \int_{-\infty}^{\infty} R(w) e^{iwt} dw \quad (2)$$

$$R(w) = \int_{-\infty}^{\infty} x(t) e^{-iwt} dt \quad (3)$$

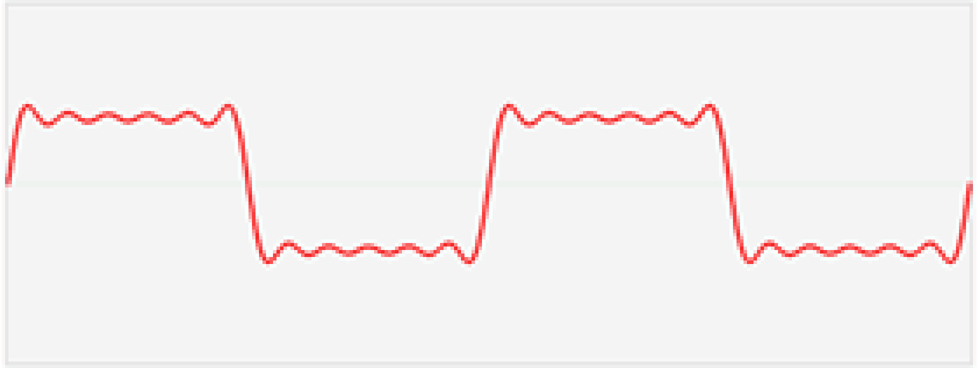
The function of angular frequency *R(w)* is the continuous Fourier transform of the function of time *x(t)*.

Fourier Transform decomposes any function into periodic functions [4].

Continuous Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} R(w) e^{iwt} dw$$

Let $x(t) = S_6(\mathrm{x})$, and $R(w) = S(f)$



The Gif was originally created by a Wikipedia user, can be found at
https://en.wikipedia.org/wiki/File:Fourier_transform_time_and_frequency_domains_(small).gif

Discrete Fourier Transform (DFT)

Discretizing the continuous Fourier Transform:

- Let $x(T_n)$ be the n^{th} element of the finite sequence $\{x_N\}$, with T as the discrete sampling interval between each data point. So $T_n = T \times n$.
- $X(w_k)$ is the DFT of $x(T_n)$ [1].

$$X(w_k) = \sum_{n=0}^{N-1} x(T_n)e^{-iw_kT_n} \quad (4)$$

with the k^{th} angular frequency $w_k = \frac{2\pi k}{NT}$

- Substitute $w_k = \frac{2\pi k}{NT}$, and $T_n = T \times n$, we have:

$$X(w_k) = \sum_{n=0}^{N-1} x(T_n)e^{-\frac{2\pi i n k}{N}} \quad (5)$$

For $0 \leq k < N$

Discrete Fourier Transform (DFT) Application

"This is the most important numerical algorithm of our lifetime..."
– Gilbert Strang (Chauvenet Prize 1977 Recipient)

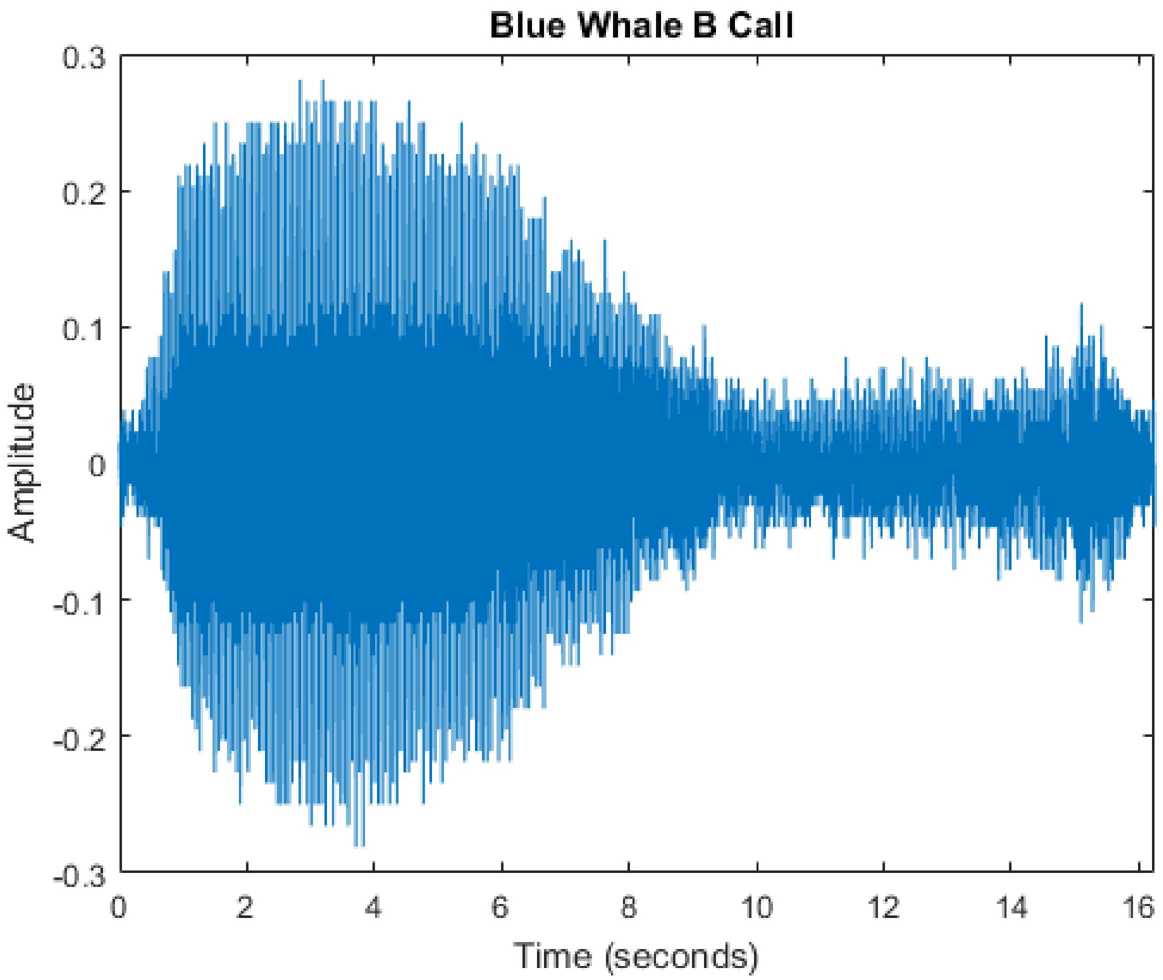
- Digital signal processing: spectral analysis of signal (human speech and hearing), Frequency Response of Systems (system analysis in frequency domain), Convolution via the Frequency Domain [8].

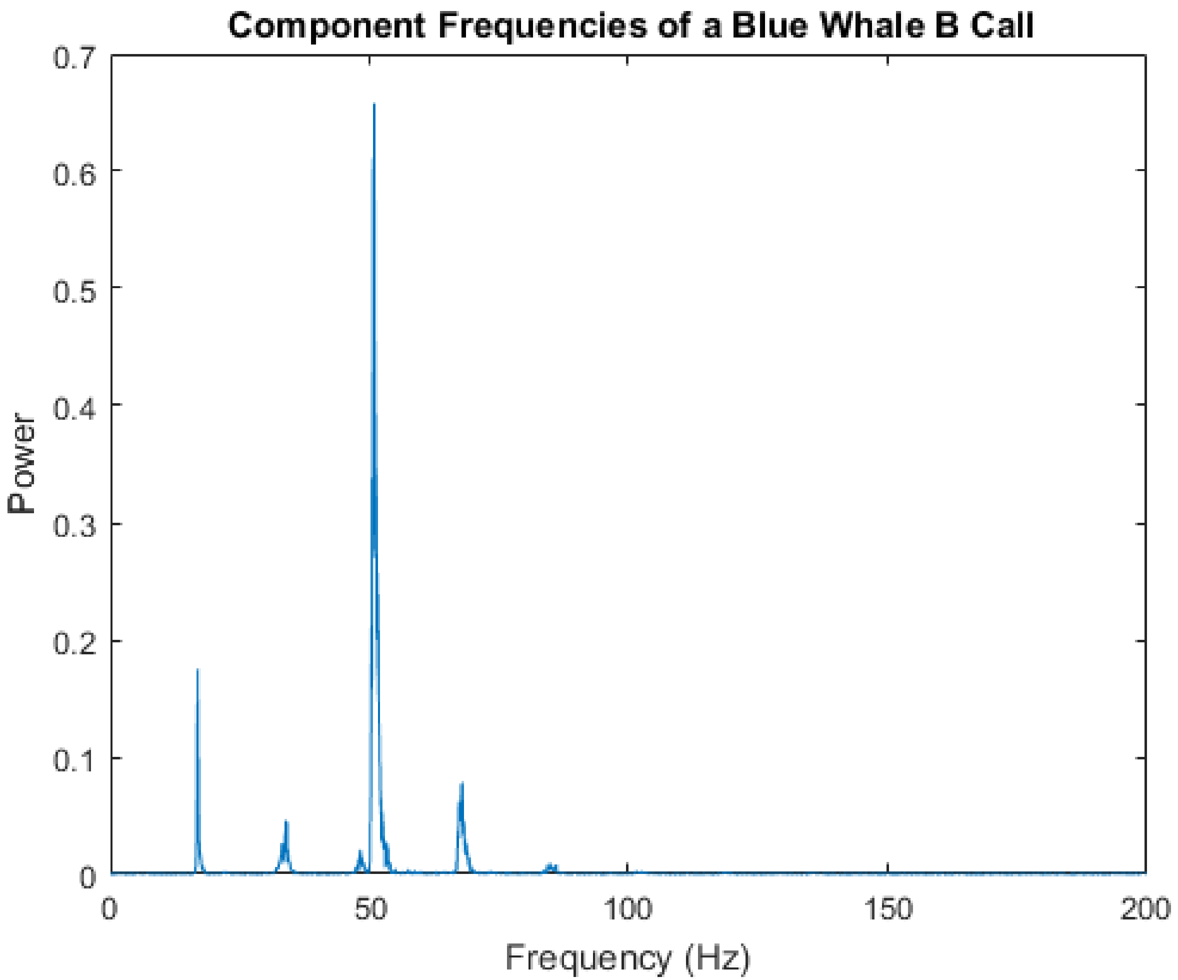
- Image processing: image analysis, image filtering, image reconstruction and image compression.

- Solving partial differential equation

Discrete Fourier Transform (DFT) Application

<u>Digital signal processing:</u> an analysis of a Blue Whale call [10]





Fast Fourier Transform (FFT)

- From (5) on page 8

$$X(w_k) = \sum_{n=0}^{N-1} x(T_n) e^{-\frac{2\pi ink}{N}}, \qquad 0 \leq k < N$$

- Let the DFT of the $x(T_n)$ data points be written as the sum of an even-indices *n=2m* as E_k and odd indices *n=2m+1* as O_k.

$$E_k = \sum_{m=0}^{N/2-1} x(T_{2m}) e^{-\frac{2\pi ik(2m)}{N}} \; and \; O_k = \sum_{m=0}^{N/2-1} x(T_{2m+1}) e^{\frac{-2\pi ik(2m)}{N}} \quad (6)$$

$$X(w_k) = \sum_{m=0}^{N/2-1} x(T_{2m}) e^{-\frac{2\pi ik(2m)}{N}} + \sum_{m=0}^{N/2-1} x(T_{2m+1}) e^{\frac{-2\pi ik(2m+1)}{N}} \quad (7)$$

$$= E_k + e^{\frac{-i2\pi k}{N}} O_k \quad (8)$$

Fast Fourier Transform (FFT)

$$X(w_k) = E_k + e^{\frac{-i2\pi k}{N}} O_k \qquad (8)$$

Since the even and odd indices DFT is periodic over *N/2*, so $E_k = E_{k\pm\frac{N}{2}}$ and $O_k = O_{k\pm\frac{N}{2}}$

We can write $X(w_k)$ such that *k* can be reduced in half, from 0 to N/2:

$$X(w_k) = E_k + e^{\frac{-i2\pi k}{N}} O_k \qquad \text{for } 0 \leq k < \frac{N}{2} \qquad (9)$$

$$X(w_k) = E_{k-\frac{N}{2}} + e^{\frac{-i2\pi k}{N}} O_{k-\frac{N}{2}} \qquad \text{for } \frac{N}{2} \leq k < N \qquad (10)$$

Also since $e^{\frac{-i2\pi(k+\frac{N}{2})}{N}} = e^{\frac{-i2\pi k}{N} - i\pi} = e^{-i\pi} e^{\frac{-i2\pi k}{N}} = -e^{\frac{-i2\pi k}{N}}$

$$X(w_{k+\frac{N}{2}}) = E_{k+\frac{N}{2}} + e^{\frac{-i2\pi\left(k+\frac{N}{2}\right)}{N}} O_{k+\frac{N}{2}} = E_k - e^{\frac{-i2\pi k}{N}} O_k \qquad (11)$$

Fast Fourier Transform (FFT)

The FFT algorithm recursively break the DFT into even and odd indices DFT E_k and O_k then calculate these smaller DFT

$$X(w_k) = E_k + e^{\frac{-i2\pi k}{N}} O_k \tag{9,10}$$

$$X(w_{k+\frac{N}{2}}) = E_k - e^{\frac{-i2\pi k}{N}} O_k \tag{11}$$

$$\text{with } 0 \leq k < \frac{N}{2}$$

Time Complexity

- "Complexity can be viewed as the maximum number of primitive operations that a program may execute. Regular operations are single additions, multiplications, assignments etc. We may leave some operations uncounted and concentrate on those that are performed the largest number of times" [2].
- Time complexity can be described in Big-O notation.

O(1): It takes the algorithm the same amount of time to compute, with different variables.

```
int a=1;
X= a+a;
```

O(N): The computation time depend linearly on variable N.

```
For (i=0, i<N, i++)
Print i;
```

O(N^2): The computation time depend on the quadratic of N.

```
For (i=0, i<N,i++)
    For(j=10, j<N+10,j++)
        print i+j;
```

O($\log_2 N$) : The computation time started with N, then get cut in half for each iteration loop

```
x=N;
Do{
   X=x/2;
      } while (x>0)
```

Time complexity of DFT (Matlab)

function output = dft(input)	+O(1)
t1= now;	+O(1)
N = length(input);	+O(1)
output = zeros(size(input));	+O(1)
for k = 0 : N - 1	+O(*N*)
s = 0;	*O(1)
for t = 0 : N - 1	+O(*N*)
s = s + input(t + 1) * exp(-2i *pi*t * k / N);	*O(1)
end	+O(1)
output(k + 1) = s;	+O(1)
end	+O(1)
t2 = now;	=O(1+1+1+1+*N**(1+*N**1+1)+1+1)
disp(t2-t1);	=O($N^2 + constant$)
end	

$$=O(N^2)$$

Code taken from http://www.nayuki.io/page/how-to-implement-the-discrete-fourier-transform

Time complexity of FFT (C++)

```
void fft(CArray& x)
{const size_t N = x.size();                       +O(1)
if (N <= 1)                                       +O(1)
   return;                                              *O(1)

CArray even = x[std::slice(0, N/2, 2)];           +O(1)
CArray  odd = x[std::slice(1, N/2, 2)];           +O(1)
fft(even);                                        +O(N log_2 N)
fft(odd);

for (size_t k = 0; k < N/2; ++k)                  +O(N)
   {   Complex t = std::polar(1.0, -2 * PI * k / N) * odd[k];   *O(1)
       x[k    ]      = even[k] + t;                     +O(1)
       x[k+N/2] = even[k] - t;                          +O(1)
   }
}
```

$$=O(N\log_2 N)$$

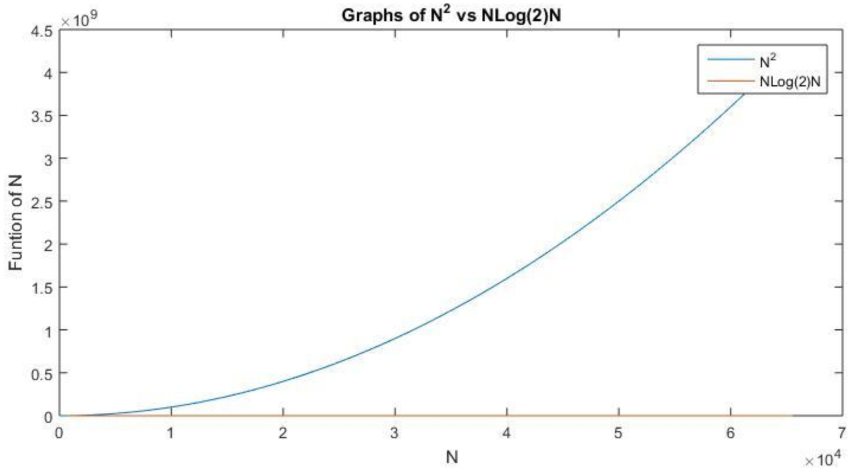
Code taken from http://rosettacode.org/wiki/Fast_Fourier_transform#C.2B.2B

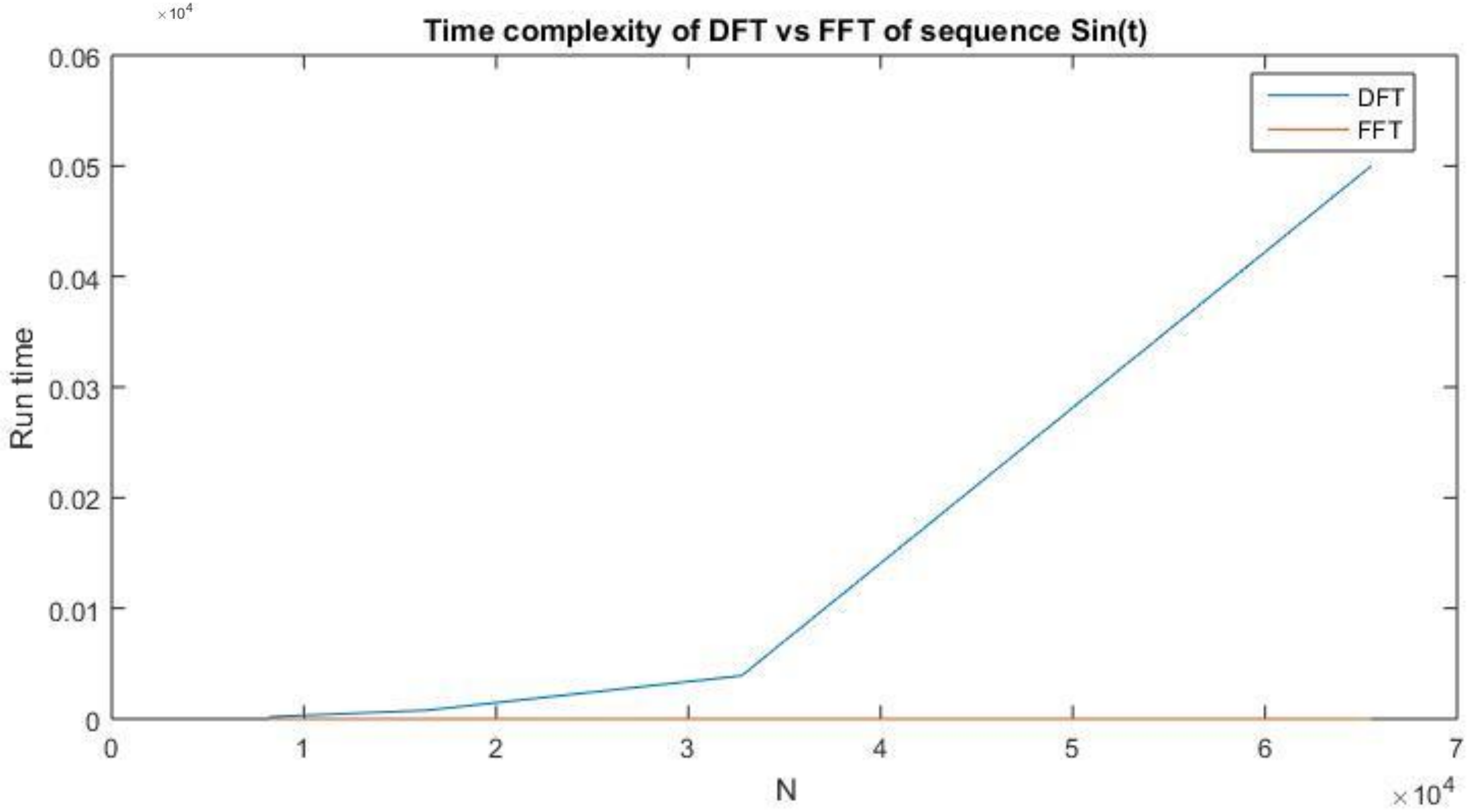
Time complexity of DFT vs. FFT

- Let $N = 2^a$, with a $\in \{13, 14, 15, 16\}$
- The time interval is from -100 to 100 with sample $t=200/N$
- The original sequence $x(t_n)$ = Sin(t_n), with n is from 0 to N-1 and $t_n = nt$

a	N $=2^a$	Run time DFT	Run time FFT
13	8192	1.88×10^{-4}	0
14	16384	7.80×10^{-4}	4.63×10^{-8}
15	32768	39×10^{-4}	4.62×10^{-8}
16	65536	50×10^{-4}	9.27×10^{-8}

Time complexity of DFT vs. FFT



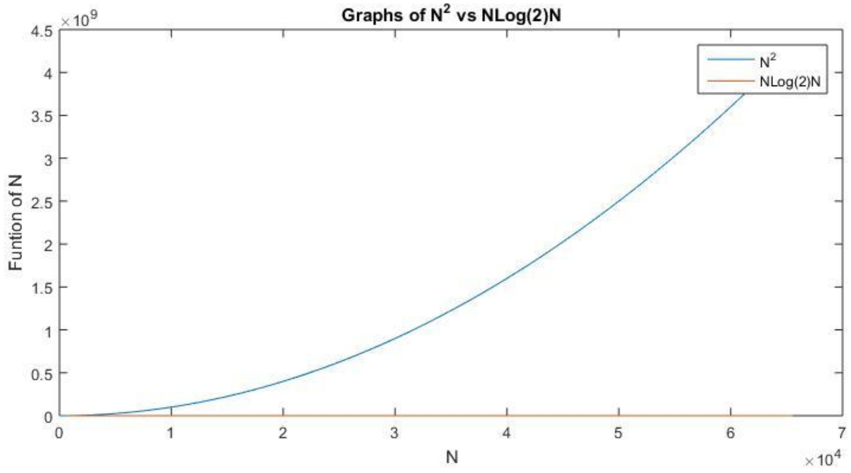


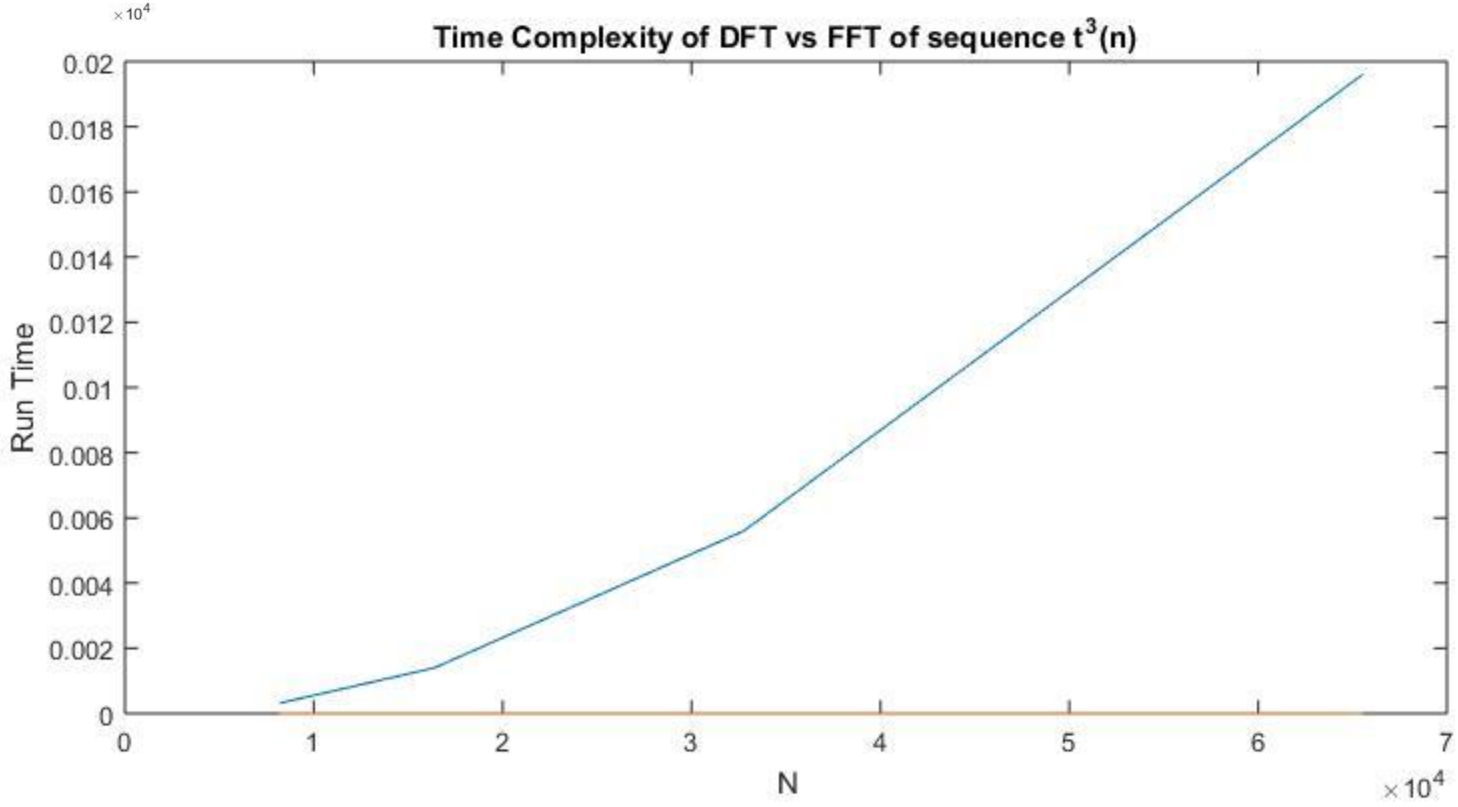
Time complexity of DFT vs. FFT

- Let $N = 2^a$, with a $\in \{13, 14, 15, 16\}$
- The time interval is from -100 to 100 with sample t=200/N
- The original sequence $x(t_n) = t_n^3$, with n is from 0 to N-1 and $t_n = nt$

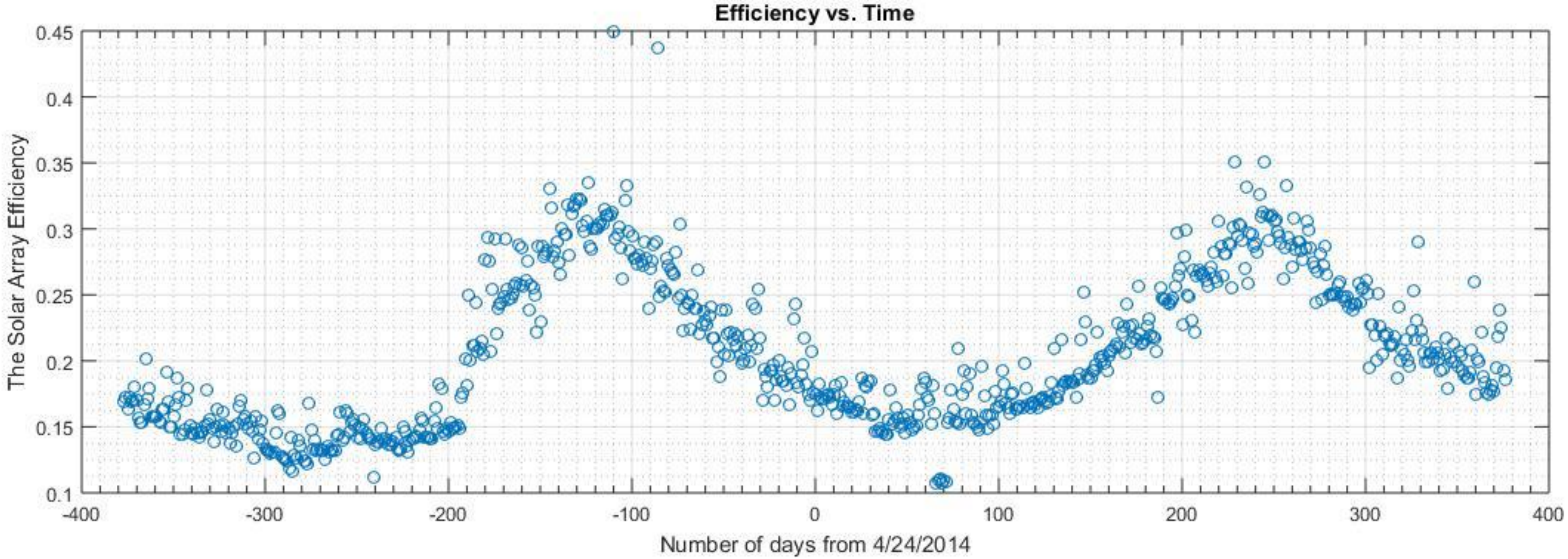
a	$N = 2^a$	Run time DFT	Run time FFT
13	8192	3.16×10^{-4}	0
14	16384	14×10^{-4}	0
15	32768	56×10^{-4}	1.15×10^{-8}
16	65536	196×10^{-4}	2.32×10^{-8}

Time complexity of DFT vs. FFT



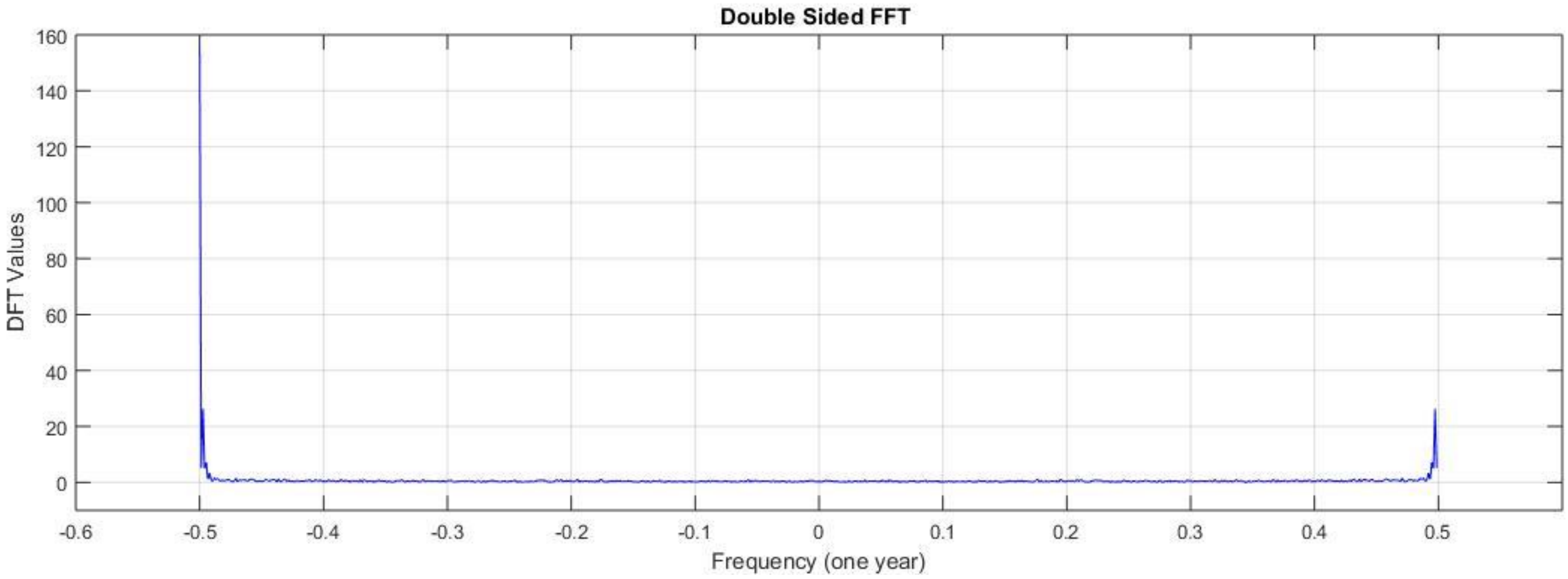


Application of FFT on Solar Array data



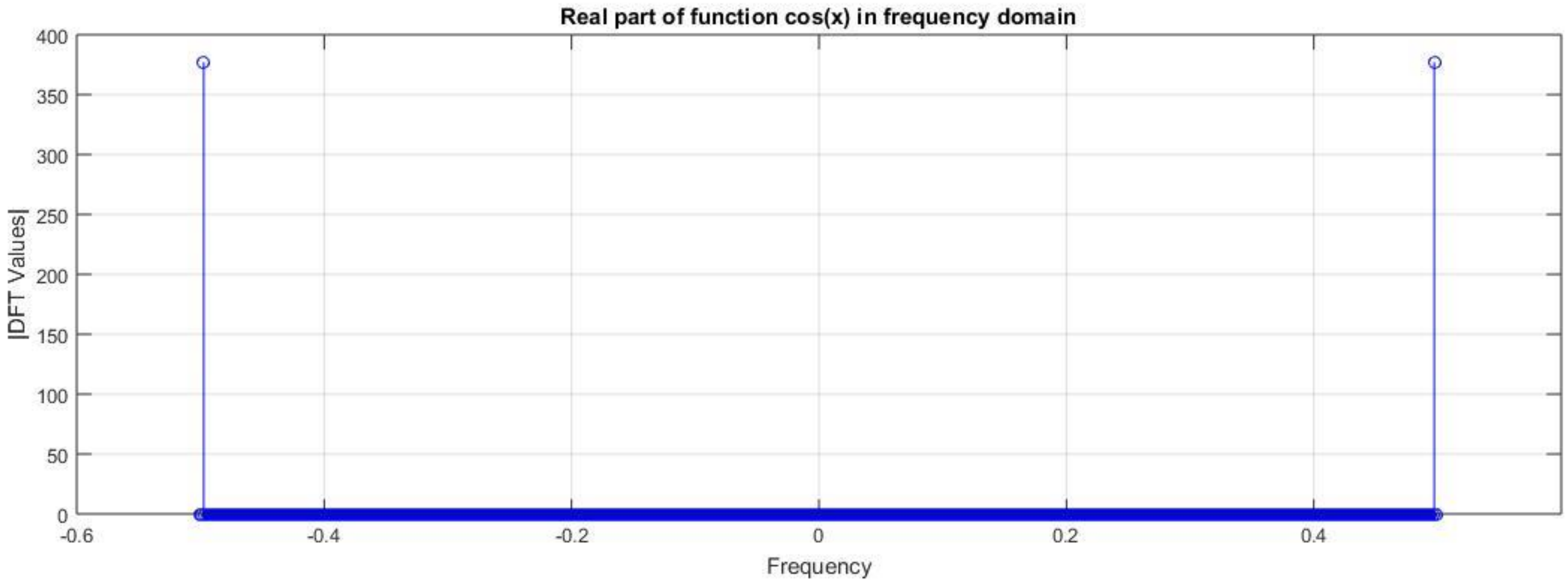
The discrete data of efficiency as a function of time looks like a periodic function.

Application of FFT on Solar Array data



The DFT of the efficiency confirms our observation that the data has a period of a year/ annually.

Application of FFT on Solar Array data



The DFT of the efficiency is similar in shape with the DFT of function Cos(x)

Conclusion

- We showed that the periodic signal of time can be represent as other primitive periodic function
- The application of the DFT in mathematics and engineering is very important as demonstrated through an example: whale signal.
- Since the DFT algorithm has a time complexity O(N^2), it is very time consuming for processing large amount of data.
- The FFT algorithm gives the same result as the DFT much faster, but with time complexity O($N\log_2 N$), allowing the run time for large amount of data to be more reasonable.

References

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