Modeling Economic Growth Using Differential Equations

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Overview

1. Introduction
2. Setting up the model
3. Solow's fundamental differential equation
4. Solving for equilibrium solutions

Introduction

- The Solow-Swan growth model was developed in 1957 by economist Robert Solow (received Nobel Prize of Economics).
- Solow's growth model is a first-order, autonomous, non-linear differential equation.
- The model includes a production function and two factors of production: capital and labor growth.

Setting up the model

The Production Function

Let *Y(t)* or Q be the annual quantity of goods produced by K units of capital and L units of labor at time t. The production function is expressed in the general form

$$Q = Y(t) = F(K(t), L(t)) \tag{1}$$

Note: Even though K and L are functions of time, we will use K instead of K(t) and L instead of L(t).

Assumptions

Assumption 1: The first important assumption of this model is that the production function, Q, is twice differentiable in capital, K and labor, L, known as the *Inada* conditions.

$$F_K(K, L) \equiv \frac{\partial F(K, L)}{\partial K} > 0 \tag{2}$$

$$F_L(K, L) \equiv \frac{\partial F(K, L)}{\partial L} > 0 \tag{3}$$

$$F_{KK}(K, L) \equiv \frac{\partial^2 F(K, L)}{\partial K^2} < 0 \tag{4}$$

$$F_{LL}(K, L) \equiv \frac{\partial^2 F(K, L)}{\partial L^2} < 0 \tag{5}$$

Assumption 2: The function F is **linearly homogeneous** of degree 1 in $K \epsilon \mathbb{R}$ and $L \epsilon \mathbb{R}$ if

$$\text{F}[\lambda\text{K(t)},\lambda\text{L(t)}]=\lambda\text{F}[\text{K(t)},\text{L(t)}] \text{ for all } \lambda \, \epsilon \, \mathbb{R}_+ \tag{6}$$

In economic terms, the production function F is said to have **constant returns to scale**.

Using assumption 2, we want to rewrite the production function in per-worker terms.
Let $\lambda = 1/L$,

$$\lambda Q = F(\lambda K, \lambda L) \tag{7}$$

$$\frac{Q}{L} = F(\frac{K}{L}, 1) \tag{8}$$

k = capital per worker and q = output per worker

$$k = \frac{K}{L} \text{ and } q = \frac{Q}{L} \tag{9}$$

$$q = f(k) \tag{10}$$

where f is defined by $f(k) = F(k,1)$.

Figure 1 shows the production function of the form $f(k) = \sqrt{k}$.

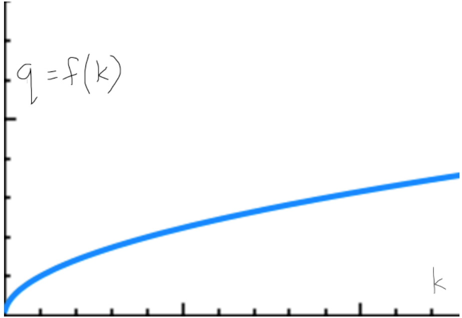

Figure 1: $q = f(k)$

rate of change of k = *marginal product of capital*

downward concavity = *diminishing marginal returns to capital*

An example of a production function of form f(k) = $\sqrt{k}$

Does f(k) = $\sqrt{k}$ satisfy the initial conditions?

f(k) = $\sqrt{k}$ = F(K,L) = $K^{1/2}L^{1/2}$

$$F_K(K,L) \equiv \frac{\partial F(K,L)}{\partial K} = \frac{1}{2}K^{-1/2}L^{1/2} > 0 \qquad \textit{Assumption 1}$$

$$F_{KK}(K,L) \equiv \frac{\partial^2 F(K,L)}{\partial K^2} = -\frac{1}{4}K^{-3/2}L^{1/2} < 0 \qquad \textit{Assumption 1}$$

$$\lambda F(K,L) = (\lambda K)^{1/2}(\lambda L)^{1/2}$$
$$\lambda F(K,L) = \lambda^{1/2}K^{1/2}\lambda^{1/2}L^{1/2}$$
$$\lambda F(K,L) = \lambda K^{1/2}L^{1/2} \qquad \textit{Assumption 2}$$

Solow's fundamental differential equation

Solow's differential equation is outlined by

$$\left\{ Rate\ of\ change\ of\ capital\ stock \right\} = \left\{ rate\ of\ investment \right\} - \left\{ rate\ of\ depreciation \right\} \tag{11}$$

and is defined by

$$\boxed{\frac{dk}{dt} = sf(k) \text{ - } \delta k} \tag{12}$$

Rate of investment:

$$i = sq = sf(k) \tag{13}$$

where s is a constant representing *savings rate*, between 0 and 1.

Rate of depreciation:

$$Annual \;\; depreciation = \delta k \tag{14}$$

where δ is a proportionality constant, referred to as *depreciation rate*.

Graphical behavior solutions

What is an equilibrium solution?

An equilibrium solution is a constant solution y=y^* to a differential equation $y' =$ f(y) such that f(y^*) $=$ 0.

Set the $\frac{dk}{dt}$ equal to 0 and solve for equilibrium solution k_s.

$$\frac{dk}{dt} = sf(k) - \delta k = 0 \tag{15}$$

$$sf(k) = \delta k \tag{16}$$

Solutions can be found at the point where δk intersects curve *sf(k)*

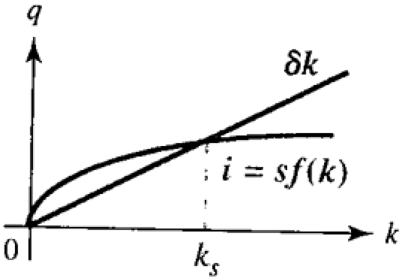

Figure 2: graphical solution

Two solutions: k=0 and k = k_s. In economics, k_s is known as the **steady-state level** of capital.

Classifying the Equilibrium Solutions

Are the equilibrium solutions asymptotically stable?

An equilibrium solution is *asymptotically stable* if all solutions with initial conditions k_0 'near' $k = k_s$ approach k_s as t approaches ∞

Figure (3) is a graph of the differential equation, where $g(k) = \frac{dk}{dt}$

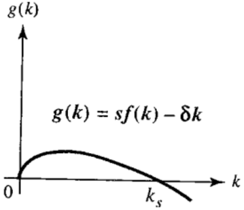

Figure 3: Graph of Solow's DE

$\frac{dk}{dt} > 0$ when $0 < k < k_s$, and $\frac{dk}{dt} < 0$ when $k > k_s$.

Figure (4) graphs the level of capital versus time.

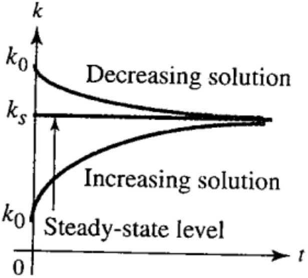

Figure 4: capital and time

k will increase towards its equilibrium level k_s if $0 < k_0 < k_s$.

k will decrease towards its equilibrium level k_s if $k_0 > k_s$.

$k = k_s$ is asymptotically stable.

Example 1: Cobb Douglas Production Function

An example of a production function:

$$Q = F(K, L) = AK^{\alpha}L^{1-\alpha}, \tag{17}$$

where $0 < \alpha < 1$, and A is another positive constant that represents *total factor productivity*.

total factor productivity = a residual that measures the output not explained by the amount of inputs.

Our production function in terms of output/per worker

$$q = \frac{Q}{L} = \frac{AK^\alpha L^{1-\alpha}}{L} = A\frac{K^\alpha L^{1-\alpha}}{L^\alpha L^{1-\alpha}} = A\left(\frac{K}{L}\right)^\alpha = Ak^\alpha \tag{18}$$

The production function is defined by

$$q = f(k) = Ak^\alpha, \quad 0 < \alpha < 1 \tag{19}$$

Now we plug equation (19) into differential equation.

$$\frac{dk}{dt} = sAk^\alpha - \delta k \tag{20}$$

Solving DE using Cobb-Douglas

Set $\frac{dk}{dt} = 0$ and solve for k_s

$$0 = sAk^{\alpha} - \delta k$$
$$0 = k^{\alpha}(sA - \delta k^{1-\alpha})$$

The equilibrium solutions are $k = 0$ and $k{=}k_s$, where

$$k_s = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{21}$$

We'll define

$$g(k) = \frac{dk}{dt} = sAk^{\alpha} - \delta\ k \tag{22}$$

Then compute

$$g'(k) = sA\alpha k^{\alpha-1} - \delta \text{ at point } k_s,$$

$$g'(k_s) = \alpha\delta - \delta < 0 \tag{23}$$

$$= \delta(\alpha - 1) < 0 \tag{24}$$

since α <1.

Solving the equation $g'(k) = 0$, we obtain the solution $k = k_i$, where

$$k_i = \left(\frac{\alpha sA}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{25}$$

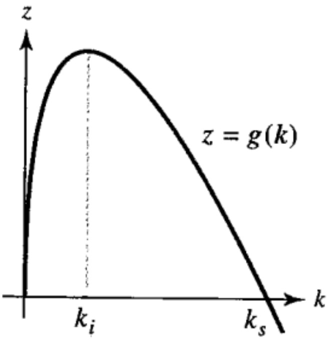

Figure 5: k vs g(k)=$\frac{dk}{dt}$

$g'(k) > 0$ when $0 < k < k_i$ and $g'(k) < 0$ when $k > k_i$.
$g(k)$ has a maximum at k_i.

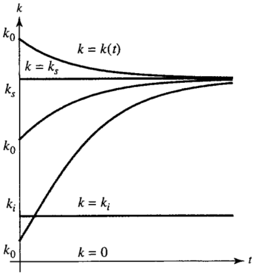

Figure 6: capital vs time

- If $k_0 > k_s$, then k is decreasing, concave upward and approaches k_s as t $\to \infty$.
- When $k_0 < k_i$, then k is increasing, and concave upwards until it reaches k_i.
- When $k_i < k_0 < k_s$, k is increasing, concave downwards, and approaches k_s as t $\to \infty$.

Solving Solow's differential equation analytically

$$\frac{dk}{dt} = \mathrm{sA}k^{\alpha} \text{ - } \delta \text{ k}$$

To solve analytically, we make a change of variable by defining

$$y = Ak^{1-\alpha} \tag{26}$$

Using the chain rule, we obtain

$$\frac{dy}{dt} = (1-\alpha)Ak^{-\alpha}\frac{dk}{dt}$$

and rewrite it as

$$\frac{dk}{dt} = \frac{1}{1-\alpha}\frac{k^{\alpha}}{A}\frac{dy}{dt}$$

Substituting this into the left-hand side of the Solow model, we obtain the equation

$$\frac{1}{1-\alpha}\frac{k^{\alpha}}{A}\frac{dy}{dt} = sAk^{\alpha} - \delta k$$

Then, dividing both sides by k^{α} and multiplying by A gives

$$\frac{1}{1-\alpha}\frac{dy}{dt} = sA^2 - \delta k^{1-\alpha} = sA^2 - \delta y$$

and simplifying yields

$$\frac{dy}{dt} = (1-\alpha)(sA^2 - \delta y) \tag{27}$$

Separation of Variables

$$\frac{dy}{dt} = (1-\alpha)(sA^2 - \delta y)$$

$$\frac{dy}{(sA^2 - \delta y)} = (1-\alpha)dt$$

$$-\frac{1}{\delta} ln(sA^2 - \delta y) = (1-\alpha)t + C$$

$$ln(sA^2 - \delta y) = -\delta(1-\alpha)t + C$$

$$sA^2 - \delta y = Ce^{-\delta(1-\alpha)t}$$

The result is

$$y = \frac{sA^2}{\delta} + Ce^{-\delta(1-\alpha)t}$$

where C is an arbitrary constant. Then replacing our y with $Ak^{1-\alpha}$ we obtain

$$Ak^{1-\alpha} = \frac{sA^2}{\delta} + Ce^{-\delta(1-\alpha)t} \tag{28}$$

Use initial condition $k(0) = k_0$ to find C.

$$C = Ak_0^{1-\alpha} - \frac{sA^2}{\delta} \tag{29}$$

Substitute C into equation (26) to obtain level of capital (k at time t.

$$k = \left[\frac{sA^2}{\delta} + \left(Ak_0^{1-\alpha} - \frac{sA^2}{\delta}\right)e^{-\delta(1-\alpha)t}\right]^{\frac{1}{1-\alpha}}. \tag{30}$$

As $t \to \infty$, the right hand side tends to zero, so $k \to k_s$.

Conclusion

- Solow's economic growth model is a great example of how we can use differential equations in real life.
- The model can be modified to include various inputs including growth in the labor force and technological improvements.
- The key to short-run growth is increased investments, while technology and efficiency improve long-run growth.

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