Stacking Blocks and Counting Permutations

Hana Mizuno

Occidental College

mizuno@oxy.edu

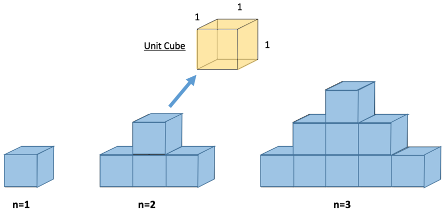
December 3rd, 2015

Overview

Background

- A father was helping out his daughter, Julia's middle school math project.
- Middle school level geometry question.
- Relationship between middle school geometry and enumeration problem (Combinatorics)

Original Question



The unit cubes are piled up in triangular form, so that the kth row has $2k - 1$ cubes. Find the **surface area** $SA(n)$ of a **pile of height** n, i.e., a pile with n rows.

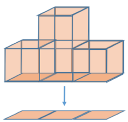
Although Julia came up with her own constructive approach to this question, we will look at Pudwell's approach.

Pudwell's Constructive Approach

Constructing a pile of height (n) from a pile of height ($n-1$) (recursive)

Pudwell's Constructive Approach

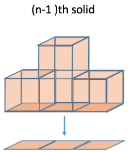
Constructing a pile of height (n) from a pile of height ($n-1$) (recursive)

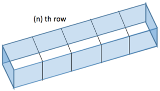


1. Separate the bottom surface of the ($n-1$)st pile

Pudwell's Constructive Approach

Constructing a pile of height (n) from a pile of height $(n-1)$ (recursive)

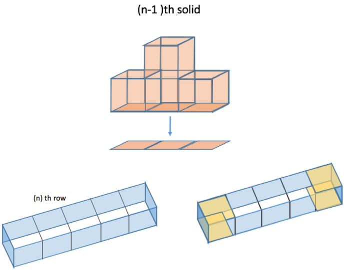




1. Separate the bottom surface of the $(n-1)$st pile
2. Construct a "ring" (row of $2n-1$ cubes without top and bottom faces) $\rightarrow$ $2(2n-1)+2 = 4n-2+2 = \mathbf{4n}$

Pudwell's Constructive Approach

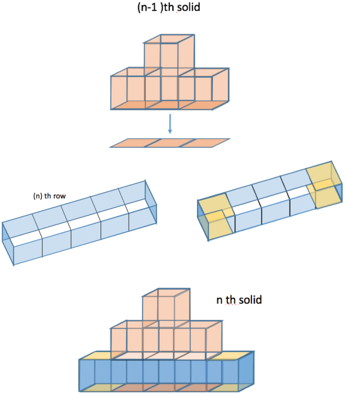
Constructing a pile of height (n) from a pile of height $(n-1)$ (recursive)



1. Separate the bottom surface of the $(n-1)$st pile
2. Construct a "ring" (row of $2n-1$ cubes without top and bottom faces) $\rightarrow$ $2(2n-1)+2 = 4n-2+2 = \mathbf{4n}$
3. Attach the top and bottoms to the two cubes at the end of the row (Yellow sides) $\rightarrow$ **4n+4**

Pudwell's Constructive Approach

Constructing a pile of height (n) from a pile of height $(n-1)$ (recursive)



1. Separate the bottom surface of the $(n-1)$st pile
2. Construct a "ring" (row of $2n-1$ cubes without top and bottom faces) $\rightarrow$ $2(2n-1)+2=4n-2+2=\mathbf{4n}$
3. Attach the top and bottoms to the two cubes at the end of the row (Yellow sides) $\rightarrow$ **4n+4**
4. glue the pile of height $(n-1)$ without a bottom face to the row of $2n-1$ cubes to form the pile of height (n).

So the surface area increases by $4n + 4$, when we go from a pile of height $(n-1)$ to a pile of height n, thus

$$SA(n) - SA(n-1) = 4n + 4 \text{ for } n \geq 2.$$

Using this recurrence and the initial condition $SA(1) = 6$, we can prove that

$$\boldsymbol{SA(n) = 2n^2 + 6n - 2} \text{ for } \boldsymbol{n \geq 1}.$$

Permutation Patterns -Definitions

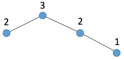
- **permutation**: "string of digits"
  example) "1224", "53928", "1212", "7948323"
- **multiset permutation**: "permutations [specifically] with more than one copy of each letter"
  example) "1221", "445599"
- **reduction**: a process that "replaces the occurrence of the *i* th smallest number with the number *i*."
  Example: Reduction of 2571165 to 2351143
  - 1s are replaced by 1
  - 2 is replaced by 2
  - 5 is replaced by 3
  - 6s are replaced by 4
  - 7 is replaced by 5

Permutation Patterns -Definitions

Assume that both p and q are permutations.

- p **contains** q: p contains q when there exists a subsequence of p that reduces to q.
  example)

  $p = 2671165$ and $q = 2321$. In this case, q is contained by p, because p has a subsequence of 6765, which can be reduced to 2321, which is equal to string q.

- p **avoids** q: p avoids q when there does not exist a subsequence of p that reduces to q.

Permutation Patterns

S_n^2: Set of permutations

- n: number of different digits in each string
- 2: indicates that each digit appears exactly twice in each string.

Example:

- S_1^2: Set of permutations with two 1s.
$$S_1^2 = \{11\}$$
- S_2^2: Set of permutations with two 1s and two 2s.
$$S_2^2 = \{1122, 1212, 1221, 2112, 2121, 2211\}$$

Let Q be a set of permutations. Let $S_n^2(Q)$ be the set of permutations that "avoids" each permutation of Q.
Example:

$$S_2^2(\{112\}) = S_2^2(112) = \{1221, 2121, 2211\}$$

Notice,

$$|S_2^2(132, 231, 2134)| = 6 = SA(1)$$
$$|S_3^2(132, 231, 2134)| = 18 = SA(2)$$

Permutation Patterns

Thus, we ask does

$$|S^2_{n+1}(132, 231, 2134)| = SA(n) \text{ for all } n \geq 1?$$

Permutation Patterns

Thus, we ask does

$$|S^2_{n+1}(132, 231, 2134)| = SA(n) \text{ for all } n \geq 1?$$

Indeed, Pudwell proved that

$$|S^2_{n+1}(132, 231, 2134)| = 2n^2 + 6n - 2 = SA(n) \text{ for all } n \geq 1.$$

$SA(n)$ and Permutation Patterns

Theorem

$|S^2_{n+1}(132, 231, 2134)| = 2n^2 + 6n - 2 = SA(n)$ *for* $n \geq 1$

$SA(n)$ and Permutation Patterns

Theorem

$|S^2_{n+1}(132, 231, 2134)| = 2n^2 + 6n - 2 = SA(n)$ *for* $n \geq 1$

- We have found that $SA(n) = SA(n-1) + 4n + 4$ for $n \geq 2$ and $SA(1) = 6$.

$SA(n)$ and Permutation Patterns

Theorem

$|S^2_{n+1}(132, 231, 2134)| = 2n^2 + 6n - 2 = SA(n)$ *for* $n \geq 1$

- We have found that $SA(n) = SA(n-1) + 4n + 4$ for $n \geq 2$ and $SA(1) = 6$.
- Now, we will find the same recurrence in Pudwell's Permutation Patterns.

$SA(n)$ and Permutation Patterns

Theorem

$|S^2_{n+1}(132, 231, 2134)| = 2n^2 + 6n - 2 = SA(n)$ *for* $n \geq 1$

- We have found that $SA(n) = SA(n-1) + 4n + 4$ for $n \geq 2$ and $SA(1) = 6$.
- Now, we will find the same recurrence in Pudwell's Permutation Patterns.
- Let $B_n = S^2_n(132, 231, 2134)$.

$$\boxed{|B_{n+1}| = |B_n| + 4n + 4, n \geq 2 \text{ and } |B_2| = 6}$$

A permutation lemma

LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

A permutation lemma

LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

Let $A_n = S_n^2(132, 231, 213)$. We begin by proving that

A permutation lemma

LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

Let $A_n = S_n^2(132, 231, 213)$. We begin by proving that

$$|A_n| = |A_{n-1}| + 2, \text{ for } n \geq 2.$$

Case of A_2 to A_3

1122	**1212**	**2121**
331122	331212	332121
313122	313212	323121
311322	312312	321321
311232	312132	321231
311223	312123	321213
133122	133212	233121
131322	132312	231321
131232	132132	231231
131223	132123	231213
113322	123312	213321
113232	123132	213231
113223	123123	213213
112332	121332	212331
112323	121323	212313
112233	121233	212133

1221	**2112**	**2211**
331221	332112	332211
313221	323112	323211
312321	321312	322311
312231	321132	322131
312213	321123	322113
133221	233112	233211
132321	231312	232311
132231	231132	232131
132213	231123	232113
123321	213312	223311
123231	213132	223131
123213	213123	223113
122331	211332	221331
122313	211323	221313
122133	211233	221133

1122

331122
313122
311322
311232
311223
133122
131322
131232
131223
113322
113232
113223
112332
112323
112233

1212

331212
313212
312312
312132
312123
133212
132312
132132
132123
123312
123132
123123
121332
121323
121233

2121

332121
323121
321321
321231
321213
233121
231321
231231
231213
213321
213231
213213
212331
212313
212133

1221

331221
313221
312321
312231
312213
133221
132321
132231
132213
123321
123231
123213
122331
122313
122133

2112

332112
323112
321312
321132
321123
233112
231312
231132
231123
213312
213132
213123
211332
211323
211233

2211

332211
323211
322311
322131
322113
233211
232311
232131
232113
223311
223131
223113
221331
221313
221133

A_2

1122	1212	2121	1221	2112	2211
331122	331212	332121	331221	332112	332211
313122	313212	323121	313221	323112	323211
311322	312312	321321	312321	321312	322311
311232	312132	321231	312231	321132	322131
311223	312123	321213	312213	321123	322113
133122	133212	233121	133221	233112	233211
131322	132312	231321	132321	231312	232311
131232	132132	231231	132231	231132	232131
131223	132123	231213	132213	231123	232113
113322	123312	213321	123321	213312	223311
113232	123132	213231	123231	213132	223131
113223	123123	213213	123213	213123	223113
112332	121332	212331	122331	211332	221331
112323	121323	212313	122313	211323	221313
112233	121233	212133	122133	211233	221133

S_3^2

1122	**1212**	**2121**
331122	331212	332121
313122	313212	323121
311322	312312	321321
311232	312132	321231
311223	312123	321213
133122	133212	233121
131322	132312	231321
131232	132132	231231
131223	132123	231213
113322	123312	213321
113232	123132	213231
113223	123123	213213
112332	121332	212331
112323	121323	212313
112233	121233	212133

1221	**2112**	**2211**
331221	332112	332211
313221	323112	323211
312321	321312	322311
312231	321132	322131
312213	321123	322113
133221	233112	233211
132321	231312	232311
132231	231132	232131
132213	231123	232113
123321	213312	223311
123231	213132	223131
123213	213123	223113
122331	211332	221331
122313	211323	221313
122133	211233	221133

Case of A_2 to A_3

1122	331122	**1212**	331212	**2121**	332121
	313122		313212		323121
	311322		312312		321321
	311232		312132		321231
	311223		312123		321213
	133122		133212		233121
	131322		132312		231321
	131232		132132		231231
	131223		132123		231213
	113322		123312		213321
	113232		123132		213231
	113223		123123		213213
	112332		121332		212331
	112323		121323		212313
	112233		121233		212133

1221	331221	**2112**	332112	**2211**	332211
	313221		323112		323211
	312321		321312		322311
	312231		321132		322131
	312213		321123		322113
	133221		233112		233211
	132321		231312		232311
	132231		231132		232131
	132213		231123		232113
	123321		213312		223311
	123231		213132		223131
	123213		213123		223113
	122331		211332		221331
	122313		211323		221313
	122133		211233		221133

Case of A_2 to A_3

1122		1212		2121	
	331122		331212		332121
	313122		313212		323121
	311322		312312		321321
	311232		312132		321231
	311223		312123		321213
	133122		133212		233121
	131322		132312		231321
	131232		132132		231231
	131223		132123		231213
	113322		123312		213321
	113232		123132		213231
	113223		123123		213213
	112332		121332		212331
	112323		121323		212313
	112233		121233		212133

1221		2112		2211	
	331221		332112		332211
	313221		323112		323211
	312321		321312		322311
	312231		321132		322131
	312213		321123		322113
	133221		233112		233211
	132321		231312		232311
	132231		231132		232131
	132213		231123		232113
	123321		213312		223311
	123231		213132		223131
	123213		213123		223113
	122331		211332		221331
	122313		211323		221313
	122133		211233		221133

Case of A_2 to A_3

1122	**1212**	**2121**
331122	331212	332121
313122	313212	323121
311322	312312	321321
311232	312132	321231
311223	312123	321213
133122	133212	233121
131322	132312	231321
131232	132132	231231
131223	132123	231213
113322	123312	213321
113232	123132	213231
113223	123123	213213
112332	121332	212331
112323	121323	212313
112233	121233	212133

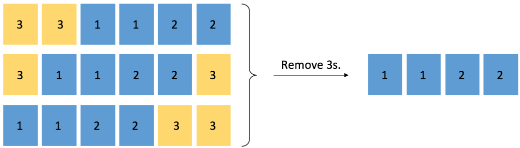
1221	**2112**	**2211**
331221	332112	332211
313221	323112	323211
312321	321312	322311
312231	321132	322131
312213	321123	322113
133221	233112	233211
132321	231312	232311
132231	231132	232131
132213	231123	232113
123321	213312	223311
123231	213132	223131
123213	213123	223113
122331	211332	221331
122313	211323	221313
122133	211233	221133

Case of A_2 to A_3

1122	1212	2121	1221	2112	2211
331122	331212	332121	331221	332112	332211
313122	313212	323121	313221	323112	323211
311322	312312	321321	312321	321312	322311
311232	312132	321231	312231	321132	322131
311223	312123	321213	312213	321123	322113
133122	133212	233121	133221	233112	233211
131322	132312	231321	132321	231312	232311
131232	132132	231231	132231	231132	232131
131223	132123	231213	132213	231123	232113
113322	123312	213321	123321	213312	223311
113232	123132	213231	123231	213132	223131
113223	123123	213213	123213	213123	223113
112332	121332	212331	122331	211332	221331
112323	121323	212313	122313	211323	221313
112233	121233	212133	122133	211233	221133

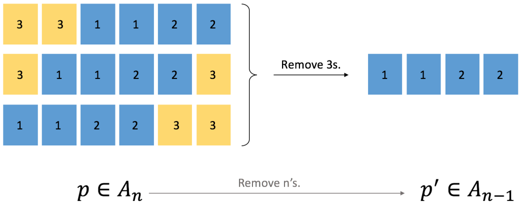
Case of A_2 to A_3

Every $p \in A_n$ can be obtained from some $p' \in A_{n-1}$



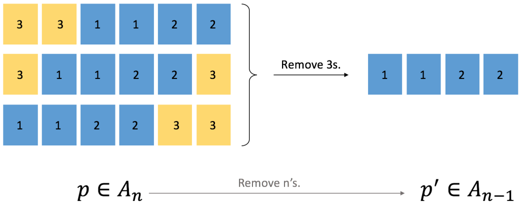
$p \in A_n \longrightarrow p' \in A_{n-1}$

Every $p \in A_n$ can be obtained from some $p' \in A_{n-1}$



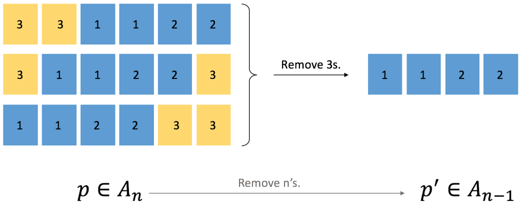
- Let $p \in A_n$, and let p' be obtained from p by removing the two copies of n contained in p.

Every $p \in A_n$ can be obtained from some $p' \in A_{n-1}$



- Let $p \in A_n$, and let p' be obtained from p by removing the two copies of n contained in p.
- Notice, $p' \in A_{n-1}$, because if $p' \notin A_{n-1}$,then some subsequence s_i of p' reduces to a sequence in $\{132, 231, 213\}$. But s_i is also a subsequence of p. This contradicts the fact that $p \in A_n$.

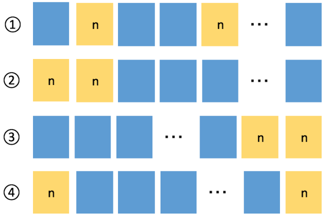
Every $p \in A_n$ can be obtained from some $p' \in A_{n-1}$



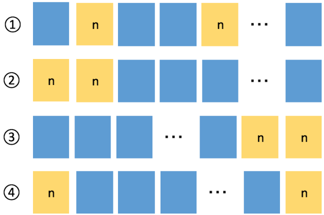
- Let $p \in A_n$, and let p' be obtained from p by removing the two copies of n contained in p.
- Notice, $p' \in A_{n-1}$, because if $p' \notin A_{n-1}$,then some subsequence s_i of p' reduces to a sequence in $\{132, 231, 213\}$. But s_i is also a subsequence of p. This contradicts the fact that $p \in A_n$.

Thus, every p_n can be obtained by adding 2 copies of n to some $p' \in A_{n-1}$.

Ways to insert two n's into p'.

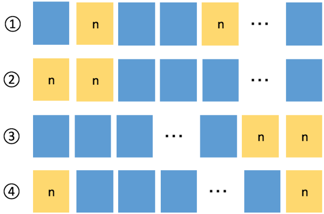


Ways to insert two n's into p'.



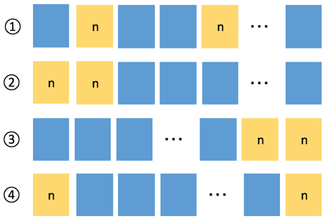
1. "between" the digits of p'

Ways to insert two n's into p'.



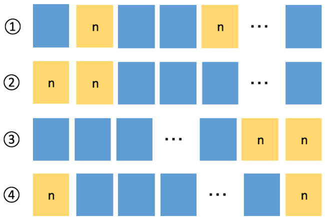
1. "between" the digits of p'
2. two n's in the beginning of p'

Ways to insert two n's into p'.



1. "between" the digits of p'
2. two n's in the beginning of p'
3. two n's at the end of p'

Ways to insert two n's into p'.



1. "between" the digits of p'
2. two n's in the beginning of p'
3. two n's at the end of p'
4. one n in the beginning and the other at the end of p'

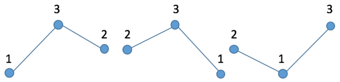
1) “between” the digits of p'

In the case where at least one n has a digit of p' to its left and a digit of p' to its right. Let a, b and c each represent a digit in p'.

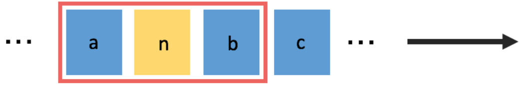
There are six different cases to consider.

1) "between" the digits of p'

Forbidden Patterns

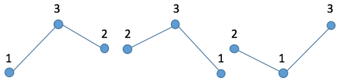


Case 1: $a < b$

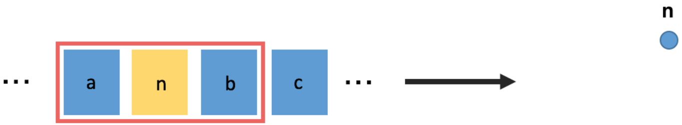


1) “between” the digits of p'

Forbidden Patterns

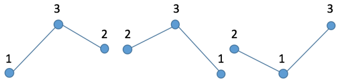


<u>Case 1: $a < b$</u>

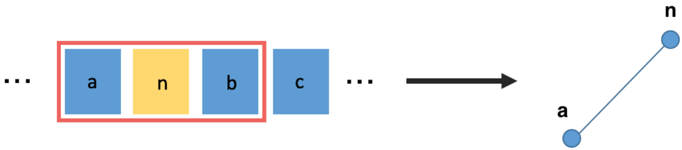


1) “between” the digits of p'

Forbidden Patterns

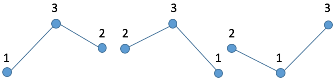


Case 1: $a < b$



1) "between" the digits of p'

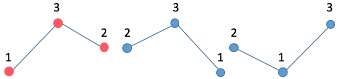
Forbidden Patterns



Case 1: $a < b$

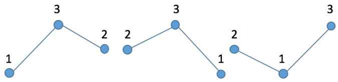
1) “between” the digits of p'

Forbidden Patterns

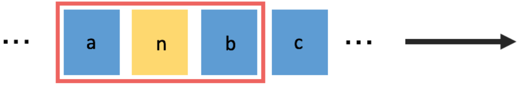


Case 1: $a < b$

Forbidden Patterns

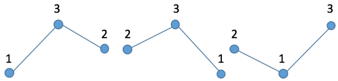


Case 2: $a > b$



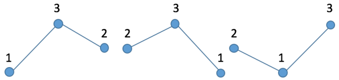
1) "between" the digits of p'

Forbidden Patterns



Case 2: $a > b$

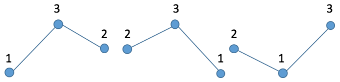
Forbidden Patterns



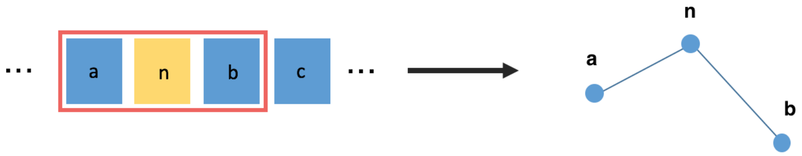
Case 2: $a > b$

1) "between" the digits of p'

Forbidden Patterns

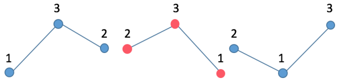


Case 2: $a > b$



1) "between" the digits of p'

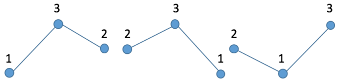
Forbidden Patterns



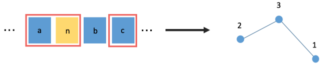
Case 2: $a > b$

1) “between” the digits of p'

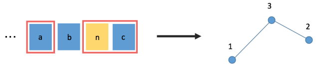
Forbidden Patterns



Case 3: $a = b > c$

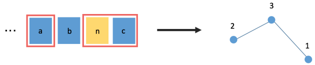


Case 5: $a < b = c$



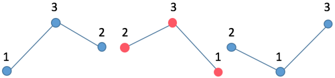
Case 4: $a = b < c$

Case 6: $a > b = c$



1) “between” the digits of p'

Forbidden Patterns

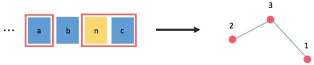


Case 3: $a = b > c$

Case 5: $a < b = c$

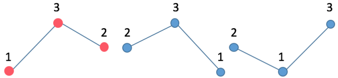
Case 4: $a = b < c$

Case 6: $a > b = c$



1) “between” the digits of p'

Forbidden Patterns

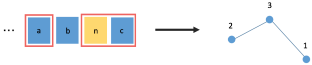


Case 3: $a = b > c$

Case 4: $a = b < c$

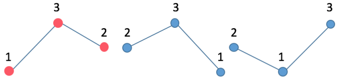
Case 5: $a < b = c$

Case 6: $a > b = c$



1) "between" the digits of p'

Forbidden Patterns

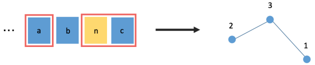


Case 3: $a = b > c$

Case 5: $a < b = c$

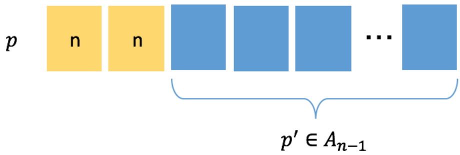
Case 4: $a = b < c$

Case 6: $a > b = c$



$\rightarrow$Never generates a member of A_n.

2) two n's in the beginning of p'



Let $p' \in A_{n-1}$ and let p be obtained from p' by placing two n's at the beginning of p'.
We show that $p \in A_n$.
There are two parts to look at:
1) Subsequences which contain n's
2) Subsequences without n's

2) two n's in the beginning of p'

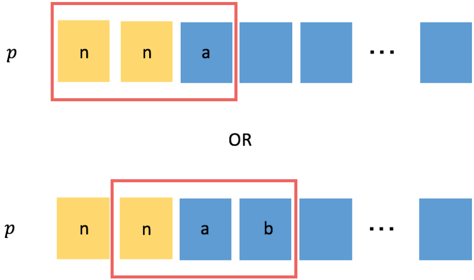
Subsequences which contain n:

Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)

2) two n's in the beginning of p'

Subsequences which contain n:

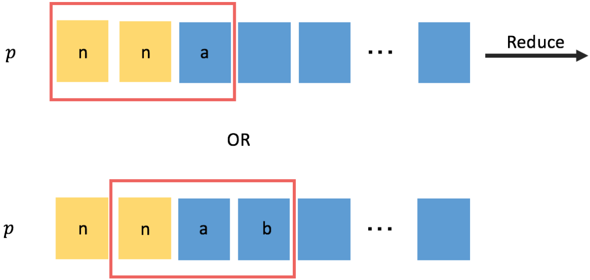
Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)



2) two n's in the beginning of p'

Subsequences which contain n:

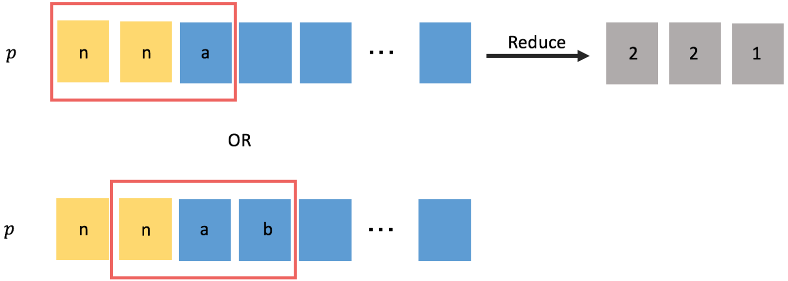
Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)



2) two n's in the beginning of p'

Subsequences which contain n:

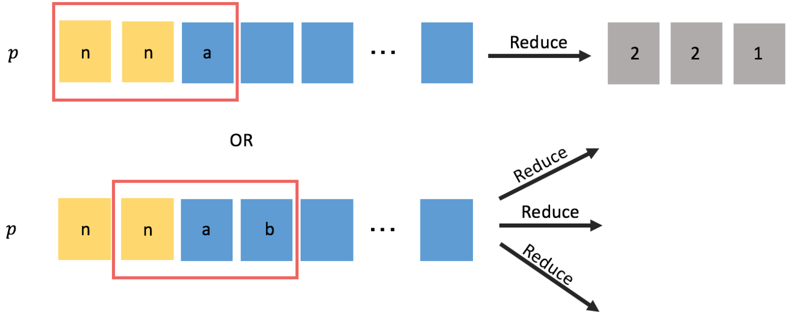
Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)



2) two n's in the beginning of p'

Subsequences which contain n:

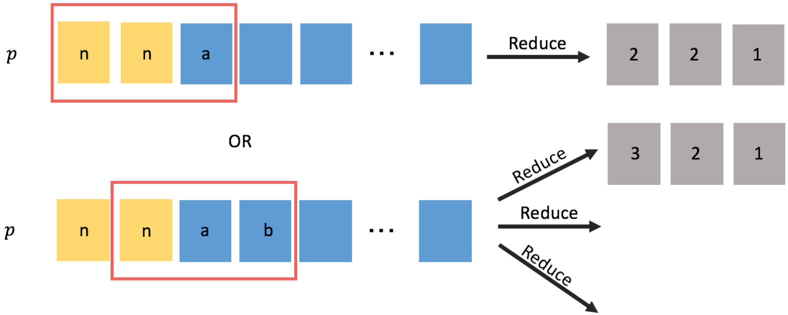
Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)



2) two n's in the beginning of p'

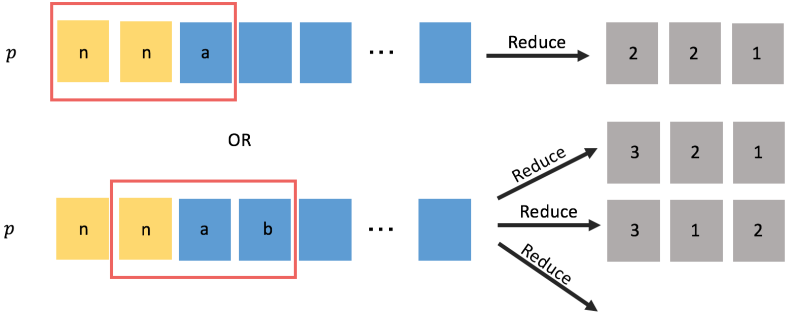
Subsequences which contain n:

Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)



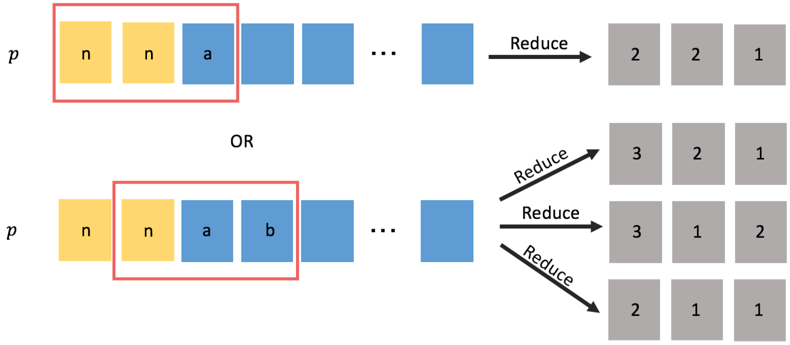
2) two n's in the beginning of p'

Subsequences which contain n:
Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)



2) two n's in the beginning of p'

Subsequences which contain n:
Any subsequence contains exactly one n will typically be reduced to a sequence r which begins with a 3, thus, $r \notin \{132, 231, 213\}$.
(A subsequence which contains both n's will be reduced to 221.)

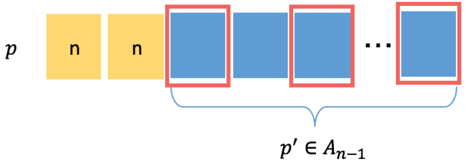


2) two n's in the beginning of p'

No subsequence which contains an n will reduce to $\{132,231,213\}$

2) two n's in the beginning of p'

Subsequences without n's:



Since $p' \in A_{n-1}$, no subsequence of p' reduces to a sequence from $\{132,231,213\}$.

2) two n's in the beginning of p'

Neither subsequences with n's nor subsequences without n's reduce to a sequence from $\{132{,}231{,}213\}$. Therefore $p \in A_n$.

$\rightarrow$ Always create a member of A_n.

3) two n's at the end of p'
4) one n in the beginning and the other at the end of p'

1122

331122
313122
311322
311232
311223
133122
131322
131232
131223
113322
113232
113223
112332
112323
112233

- These cases will create members of A_n only when all of the digits of p' are in nondecreasing order.
- Both cases have at least one n at the end of p'.
- The only forbidden pattens with the biggst digit at the end is 213.
- If p' is not in nondecreasing order, then there will always be 21 subsequence in p', which results in generating 213 subsequence in p.

- For each $p' \in A_{n-1}$, placing two n's at the beginning of p' generates a $p \in A_n$.

1122		**1212**		**2121**	
	331122		**331212**		**332121**
	313122		313212		323121
	311322		312312		321321
	311232		312132		321231
	311223		312123		321213
	133122		133212		233121
	131322		132312		231321
	131232		132132		231231
	131223		132123		231213
	113322		123312		213321
	113232		123132		213231
	113223		123123		213213
	112332		121332		212331
	112323		121323		212313
	112233		121233		212133

1221		**2112**		**2211**	
	331221		**332112**		**332211**
	313221		323112		323211
	312321		321312		322311
	312231		321132		322131
	312213		321123		322113
	133221		233112		233211
	132321		231312		232311
	132231		231132		232131
	132213		231123		232113
	123321		213312		223311
	123231		213132		223131
	123213		213123		223113
	122331		211332		221331
	122313		211323		221313
	122133		211233		221133

- For each $p' \in A_{n-1}$, placing two n's at the beginning of p' generates a $p \in A_n$.
- There is exactly one permutation $p' \in A_{n-1}$ whose digits are in nondecreasing order. Placing two n's at the end of p' or "surrounding" p' with one n on each side generates two more permutations from A_n.

1122	331122	**1212**	331212	**2121**	332121
	313122		313212		323121
	311322		312312		321321
	311232		312132		321231
	311223		312123		321213
	133122		133212		233121
	131322		132312		231321
	131232		132132		231231
	131223		132123		231213
	113322		123312		213321
	113232		123132		213231
	113223		123123		213213
	112332		121332		212331
	112323		121323		212313
	112233		121233		212133
1221	331221	**2112**	332112	**2211**	332211
	313221		323112		323211
	312321		321312		322311
	312231		321132		322131
	312213		321123		322113
	133221		233112		233211
	132321		231312		232311
	132231		231132		232131
	132213		231123		232113
	123321		213312		223311
	123231		213132		223131
	123213		213123		223113
	122331		211332		221331
	122313		211323		221313
	122133		211233		221133

Therefore,

$$|A_n| = |A_{n-1}| + 2$$

We have seen that $A_2 = 6$. Therefore,
$|A_3| = 6 + 2 = 8$ and $|A_4| = 8 + 2 = 10$.
This indicates that $|A_n|$ grows linearly, so

$$|A_n| = 2n + 2, \text{ for } n \geq 2$$

Recurrence in Pudwell's Permutation

Let's look at the $B_n = S_n^2(132, 231, 2134)$.
Similar to $A_n = S_n^2(132, 231, 213)$, each $q \in B_{n+1}$ can be generated by inserting two copies of (n+1) into some $q' \in B_n$.

1. "between" the digits of q'
2. two $(n+1)$'s in the beginning of q'
3. two $(n+1)$'s at the end of q'
4. one $(n+1)$ in the beginning and the other at the end of q'

Let $A_n = S_n^2(132, 231, 213)$ and let $B_n = S_n^2(132, 231, 2134)$.

A3
331122
331212
331221
332112
332121
332211
311223
112233

B3		
331122	112233	311223
331212	121233	312123
331221	122133	312213
332112	211233	321123
332121	212133	321213
332211	221133	322113

A4
44331122
44331212
44331221
44332112
44332121
44332211
44311223
44112233
41122334
11223344

Let $A_n = S_n^2(132, 231, 213)$ and let $B_n = S_n^2(132, 231, 2134)$.

A3

331122
331212
331221
332112
332121
332211
311223
112233

B3

331122	112233	311223
331212	121233	312123
331221	122133	312213
332112	211233	321123
332121	212133	321213
332211	221133	322113

"44"+ B3

A4

44331122
44331212
44331221
44332112
44332121
44332211
44311223
44112233
41122334
11223344

B4

44331122
44331212
44331221
44332112
44332121
44332211
44112233
44121233
44122133
44211233
44212133
44221133
44311223
44312123
44312213
44321123
44321213
44322113

Let $A_n = S_n^2(132, 231, 213)$ and let $B_n = S_n^2(132, 231, 2134)$.

A3

331122
331212
331221
332112
332121
332211
311223
112233

B3

331122	112233	311223
331212	121233	312123
331221	122133	312213
332112	211233	321123
332121	212133	321213
332211	221133	322113

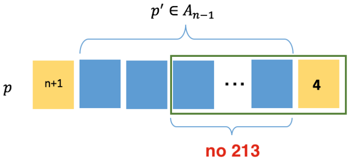
A3 + "44"

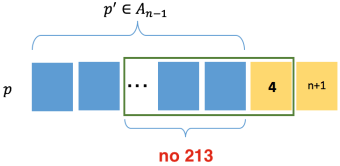
A4

44331122
44331212
44331221
44332112
44332121
44332211
44311223
44112233
41122334
11223344

B4

44331122	33112244
44331212	33121244
44331221	33122144
44332112	33211244
44332121	33212144
44332211	33221144
44112233	31122344
44121233	11223344
44122133	
44211233	
44212133	
44221133	
44311223	
44312123	
44312213	
44321123	
44321213	
44322113	





Let $A_n = S_n^2(132, 231, 213)$ and let $B_n = S_n^2(132, 231, 2134)$.

A3

331122
331212
331221
332112
332121
332211
311223
112233

B3

331122	112233	311223
331212	121233	312123
331221	122133	312213
332112	211233	321123
332121	212133	321213
332211	221133	322113

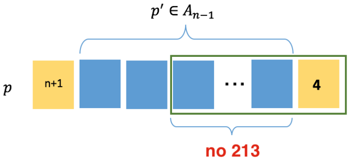
A4

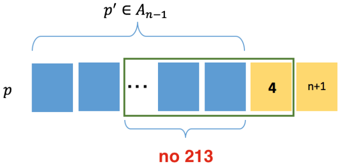
44331122
44331212
44331221
44332112
44332121
44332211
44311223
44112233
41122334
11223344

“4”+ A3 +”4”

B4

44331122	33112244	43311224
44331212	33121244	43312124
44331221	33122144	43312214
44332112	33211244	43321124
44332121	33212144	43321214
44332211	33221144	43322114
44112233	31122344	43112234
44121233	11223344	41122334
44122133		
44211233		
44212133		
44221133		
44311223		
44312123		
44312213		
44321123		
44321213		
44322113		





Let $A_n = S_n^2(132, 231, 213)$ and let $B_n = S_n^2(132, 231, 2134)$.

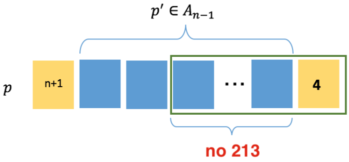
A3
331122
331212
331221
332112
332121
332211
311223
112233

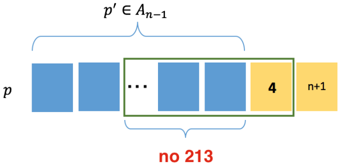
B3		
331122	112233	311223
331212	121233	312123
331221	122133	312213
332112	211233	321123
332121	212133	321213
332211	221133	322113

"4"+ A3 +"4"

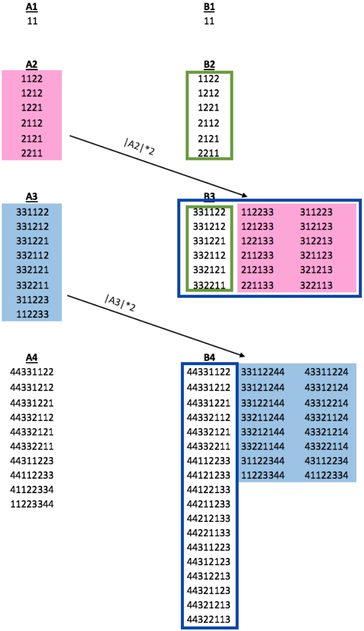
A4
44331122
44331212
44331221
44332112
44332121
44332211
44311223
44112233
41122334
11223344

B4		
44331122	33112244	43311224
44331212	33121244	43312124
44331221	33122144	43312214
44332112	33211244	43321124
44332121	33212144	43321214
44332211	33221144	43322114
44112233	31122344	43112234
44121233	11223344	41122334
44122133		
44211233		
44212133		
44221133		
44311223		
44312123		
44312213		
44321123		
44321213		
44322113		

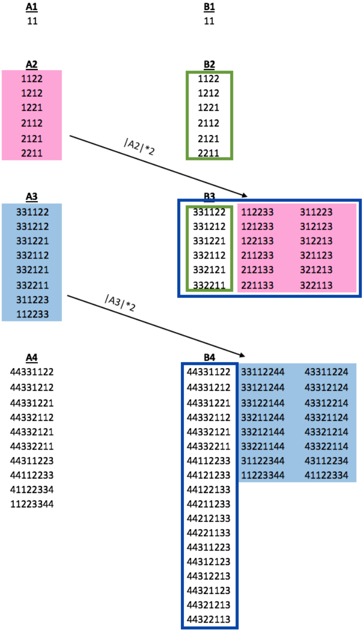




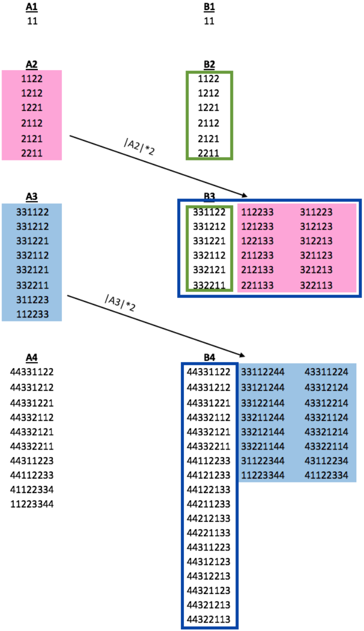
$$|B_4| = |B_3| + 2|A_3|$$


A1
11
B1
11
A2
1122
1212
1221
2112
2121
2211
B2
1122
1212
1221
2112
2121
2211
|A2|*2
A3
331122
331212
331221
332112
332121
332211
311223
112233
B3
331122 112233 311223
331212 121233 312123
331221 122133 312213
332112 211233 321123
332121 212133 321213
332211 221133 322113
|A3|*2
A4
44331122
44331212
44331221
44332112
44332121
44332211
44311223
44112233
41122334
11223344
B4
44331122 33112244 43311224
44331212 33121244 43312124
44331221 33122144 43312214
44332112 33211244 43321124
44332121 33212144 43321214
44332211 33221144 43322114
44112233 31122344 43112234
44121233 11223344 41122334
44122133
44211233
44212133
44221133
44311223
44312123
44312213
44321123
44321213
44322113

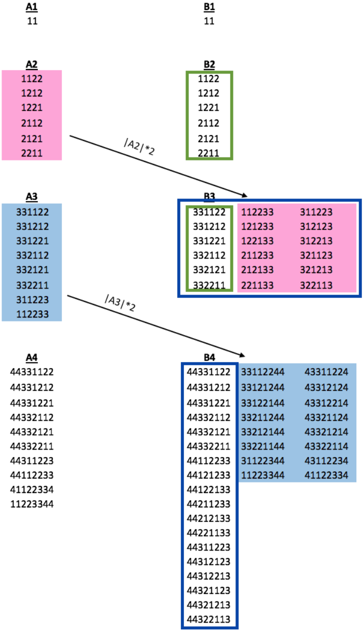
- $|B_4| = |B_3| + 2|A_3|$



- $|B_4| = |B_3| + 2|A_3|$
- $|B_{n+1}| = |B_n| + 2|A_n|$



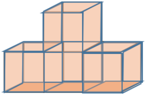
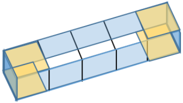
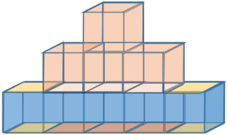
- $|B_4| = |B_3| + 2|A_3|$
- $|B_{n+1}| = |B_n| + 2|A_n|$
- $|A_n| = 2n + 2$
  (From Lemma 1)



- $|B_4| = |B_3| + 2|A_3|$
- $|B_{n+1}| = |B_n| + 2|A_n|$
- $|A_n| = 2n + 2$
  (From Lemma 1)

$$\boxed{|B_{n+1}| = |B_n| + 4n + 4, \; |B_2| = 6}$$

Recurrence

 +  = 

$SA(n-1) + 4n+4 = SA(n)$

$|B_n| + 2|A_n| = |B_{n+1}|$

$|B_n| + 2(2n+2) = |B_{n+1}|$

Theorem

$|S^2_{n+1}(132, 231, 2134)| = 2n^2 + 6n - 2 = SA(n)$ *for* $n \geq 1$

- $SA(n) = SA(n-1) + 4n + 4$ for $n \geq 2$ and $SA(1) = 6$.
- $B_n = S^2_n(132, 231, 2134)$.
- $|B_{n+1}| = |B_n| + 4n + 4$, $n \geq 2$ and $|B_2| = 6$

Conclusion

- Relationship between middle school geometry and Combinatorics
- Application of discrete math to geometry question
- This is just a part of Pudwell's study on permutation that avoids other permutations, so it would be interesting to read and investigate other Enumeration of Words with Forbidden Patterns studies.

References

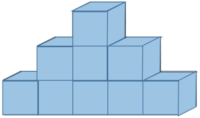
- PUDWELL, LARA K.,"*Stacking Blocks and Counting Permutations*",*Mathematics Magazine*,83.4,(2008),297-302.
- BURSTEIN, ALEXANDER,"*Enumeration of Words with Forbidden Patterns*", Dissertation, University of Pennsylvania, 1998.

Acknowledgements

- Professor Sundberg
- Professor Buckmire
- Megan Liu
- Kristin Oberiano
- all of my friends

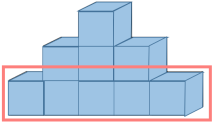
Backup

Original Solution



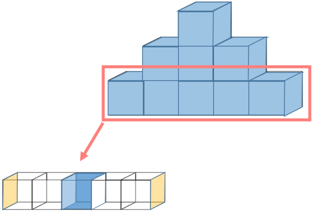
- a pile of height $(n-1)$ glued together with a row of **(2n-1)**cubes

Original Solution



- a pile of height $(n-1)$ glued together with a row of **(2n-1)**cubes

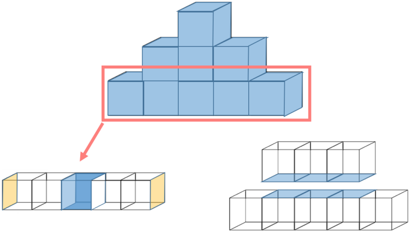
Original Solution



- a pile of height $(n-1)$ glued together with a row of **(2n-1)**cubes

$$4(2n-1)+2=8n-2$$

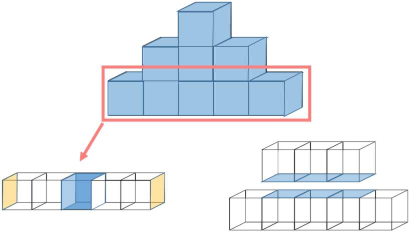
Original Solution



- a pile of height $(n-1)$ glued together with a row of **(2n-1)**cubes
$$4(2n-1)+2=8n-2$$

- $2n-3$ sides of cubes overlap
$$(8n-2)-2(2n-3)=4n+4$$

Original Solution



- a pile of height $(n-1)$ glued together with a row of **(2n-1)**cubes
  $$4(2n-1)+2=8n-2$$
- $2n-3$ sides of cubes overlap
  $$(8n-2)-2(2n-3)=4n+4$$

So the surface area increases by $4n+4$, when we go from a pile of height $(n-1)$ to a pile of height n, thus

$$SA(n)-SA(n-1)=4n+4 \text{ for } n \geq 2.$$

Using this recurrence and the initial condition $SA(1)=6$, we can prove that

$$\boldsymbol{SA(n)=2n^2+6n-2 \textbf{ for } n \geq 1}.$$

Original Solution (Inductive Proof)

Proof by Induction:
Prove: $SA(n) = 2n^2 + 6n - 2$ for $n \geq 1$

Basis: n $= 1$

$$\begin{aligned} 2(1)^2 + 6(1) - 2 &= 2 + 6 - 2 \\ &= 6 \\ &= SA(1)\checkmark \end{aligned}$$

Original Solution (Inductive Proof)

Let's apply our recurrence ($SA(n) - SA(n-1) = 4n + 4$) to $SA(n+1) - SA(n)$,
i.e.,

$$SA(n+1) - SA(n) = 4(n+1) + 4,$$

thus,

$$SA(n+1) = SA(n) + 4(n+1) + 4.$$

By induction, $SA(n) = 2n^2 + 6n - 2$, thus,

$$\begin{aligned} SA(n+1) &= 2n^2 + 6n - 2 + (4(n+1) + 4) \\ &= 2n^2 + 6n - 2 + 4n + 4 + 4 \\ &= 2n^2 + 4n + 2 + 6n + 6 - 2 \\ &= 2(n+1)^2 + 6(n+1) - 2 \end{aligned}$$

therefore, by the Principle of Mathematical Induction, $SA(n) = 2n^2 + 6n - 2$ for $n \geq 1$.

Permutation Patterns

For example, $S_n^2(\{132, 231, 2134\}) = S_n^2(132, 231, 2134)$ will be:
When n=2

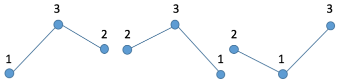
$$S_2^2(132, 231, 2134) = \{1122, 1212, 1221, 2112, 2121, 2211\}$$

When n=3

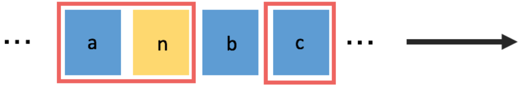
$$\begin{aligned} S_3^2(132, 231, 2134) = \{&112233, 121233, 122133, 211233, 212133,\\ &221133, 311223, 312123, 312213, 321123, 332211\\ &321213, 322113, 331122, 331212, 331221, 332112\\ &332121\} \end{aligned}$$

1) “between” the digits of p'

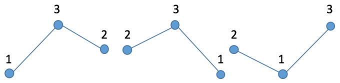
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



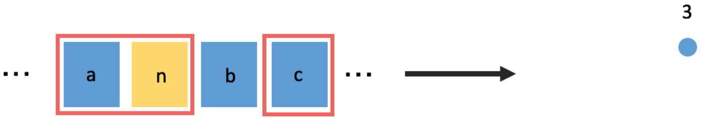
Case 3: $a = b > c$



LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

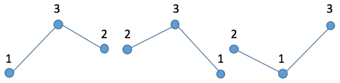


Case 3: $a = b > c$

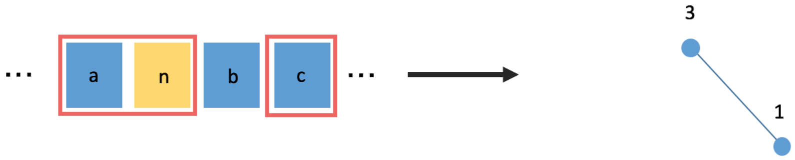


1) "between" the digits of p'

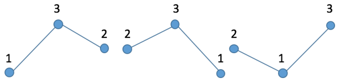
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



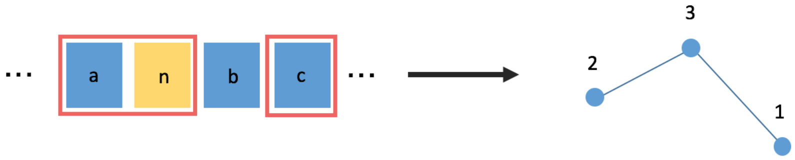
Case 3: $a = b > c$



LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

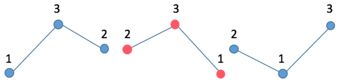


Case 3: $a = b > c$

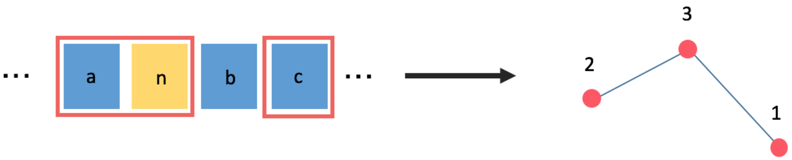


1) "between" the digits of p'

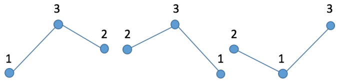
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



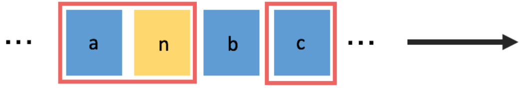
<u>Case 3: $a = b > c$</u>



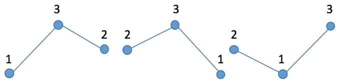
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



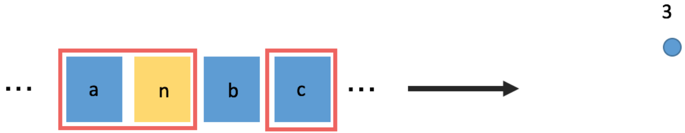
Case 4: $a = b < c$



LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

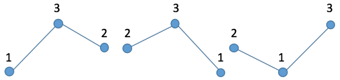


Case 4: $a = b < c$

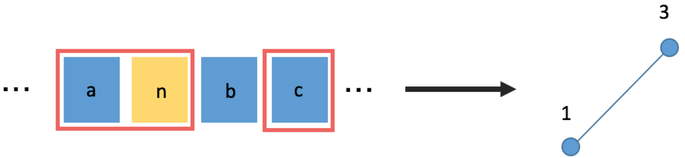


1) "between" the digits of p'

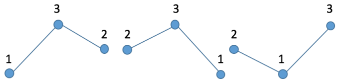
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



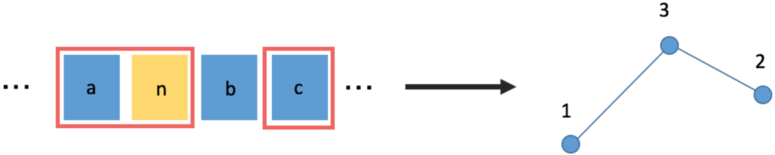
Case 4: $a = b < c$



LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

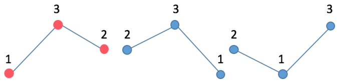


Case 4: $a = b < c$

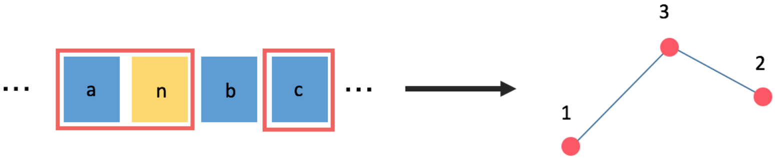


1) "between" the digits of p'

LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

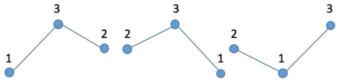


Case 4: $a = b < c$

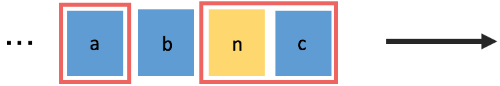


1) "between" the digits of p'

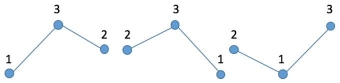
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



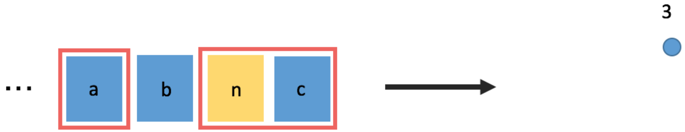
Case 5: $a < b = c$



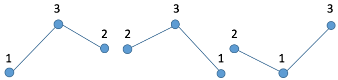
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



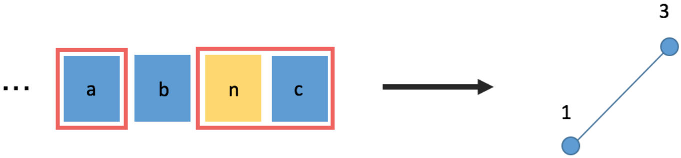
Case 5: $a < b = c$



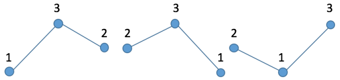
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



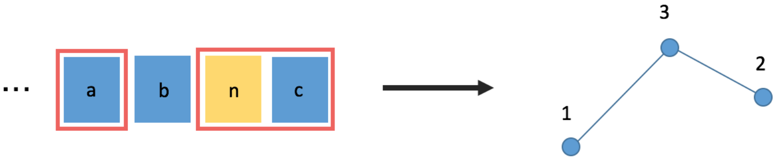
Case 5: $a < b = c$



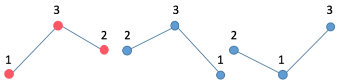
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



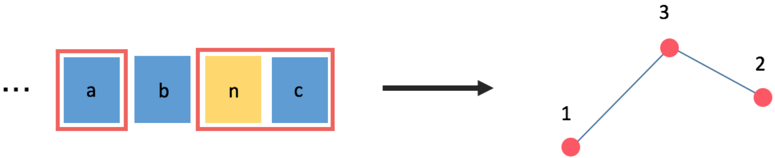
Case 5: $a < b = c$



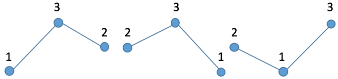
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



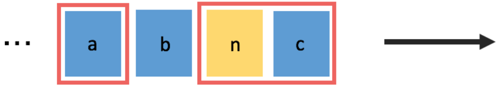
Case 5: $a < b = c$



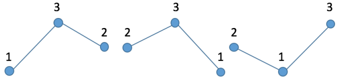
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



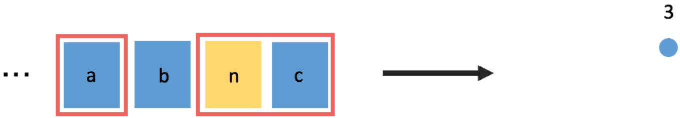
Case 6: $a > b = c$



LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

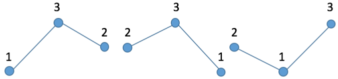


Case 6: $a > b = c$

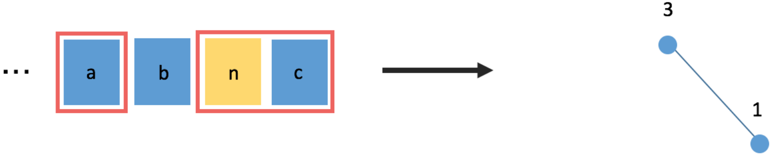


1) “between” the digits of p'

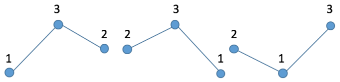
LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



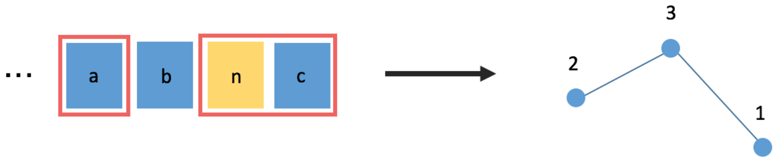
<u>Case 6: $a > b = c$</u>



LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

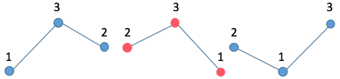


Case 6: $a > b = c$

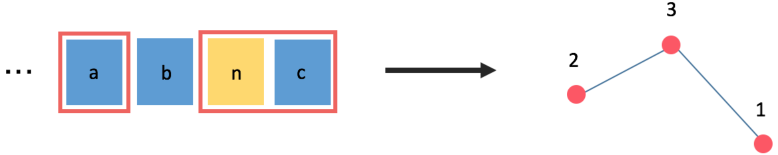


1) "between" the digits of p'

LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$

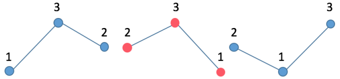


Case 6: $a > b = c$

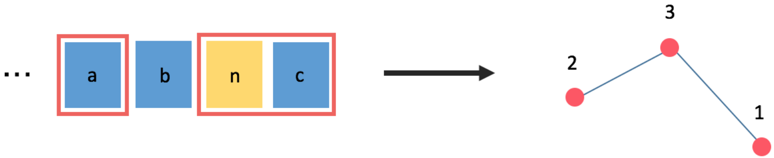


1) "between" the digits of p'

LEMMA 1: $|S_n^2(132, 231, 213)| = 2n + 2$ for $n \geq 2$



Case 6: $a > b = c$



→

All the permutations is avoided by the forbitten patterns.

3) two n's at the end of p'

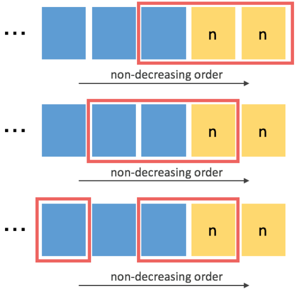
There are two different parts to look at:

a) Subsequences with n's.

3) two n's at the end of p'

There are two different parts to look at:

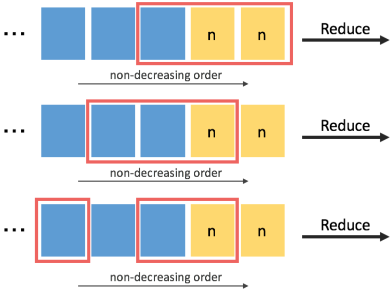
a) Subsequences with n's.



3) two n's at the end of p'

There are two different parts to look at:

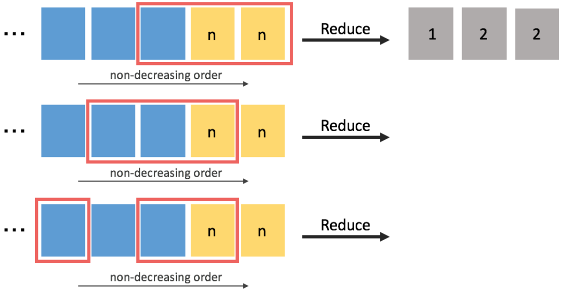
a) Subsequences with n's.



3) two n's at the end of p'

There are two different parts to look at:

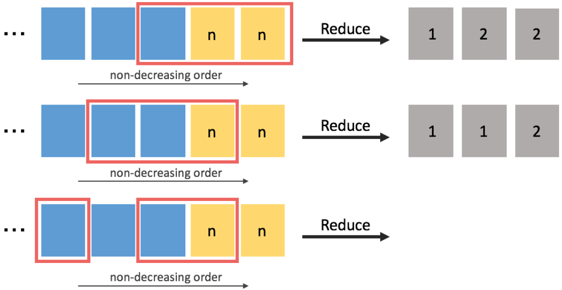
a) Subsequences with n's.



3) two n's at the end of p'

There are two different parts to look at:

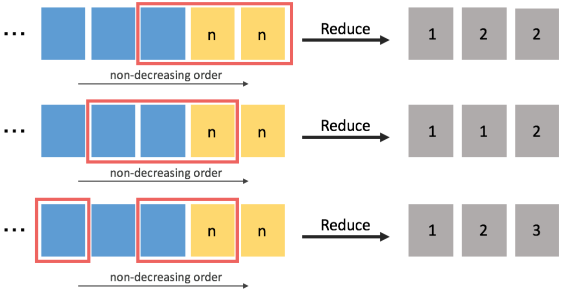
a) Subsequences with n's.



3) two n's at the end of p'

There are two different parts to look at:

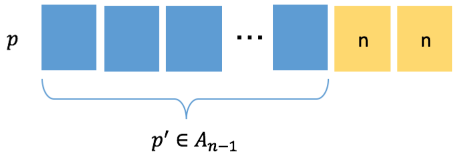
a) Subsequences with n's.



3) two n's at the end of p'

There are two different parts to look at:

b) Subsequences without n's.
Because $p' \in A_{n-1}$, no subsequence of p' reduces to a sequence from $\{132,231,213\}$.



$\rightarrow$ Create member of A_n.

4) one n in the beginning and one at the end of p'

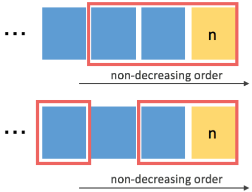
There are 3 parts to look at:
a) Subsequences end with n.
n is the biggest digit in the permutation, so every single subsequences ends with 3. The only sequence in $\{132,231,213\}$ that ends with 3 is 213. However, the permutation has all the digits placed in increasing order; thereofre, it is impossible to have any subsequences reduce to 213.

4) one n in the beginning and one at the end of p'

There are 3 parts to look at:

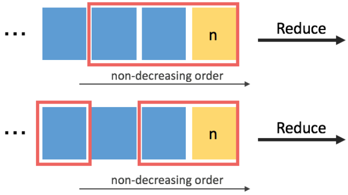
a) Subsequences end with n.

n is the biggest digit in the permutation, so every single subsequences ends with 3. The only sequence in $\{132,231,213\}$ that ends with 3 is 213. However, the permutation has all the digits placed in increasing order; thereofre, it is impossible to have any subsequences reduce to 213.



4) one n in the beginning and one at the end of p'

There are 3 parts to look at:
a) Subsequences end with n.
n is the biggest digit in the permutation, so every single subsequences ends with 3. The only sequence in {132,231,213} that ends with 3 is 213. However, the permutation has all the digits placed in increasing order; thereofre, it is impossible to have any subsequences reduce to 213.

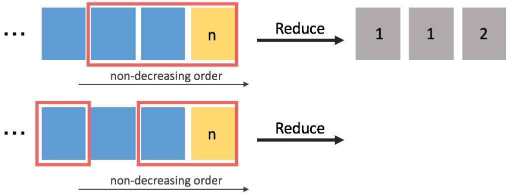


4) one n in the beginning and one at the end of p'

There are 3 parts to look at:

a) Subsequences end with n.

n is the biggest digit in the permutation, so every single subsequences ends with 3. The only sequence in {132,231,213} that ends with 3 is 213. However, the permutation has all the digits placed in increasing order; thereofre, it is impossible to have any subsequences reduce to 213.

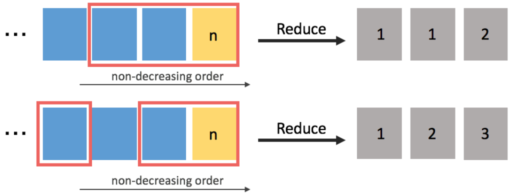


4) one n in the beginning and one at the end of p'

There are 3 parts to look at:

a) Subsequences end with n.

n is the biggest digit in the permutation, so every single subsequences ends with 3. The only sequence in {132,231,213} that ends with 3 is 213. However, the permutation has all the digits placed in increasing order; thereofre, it is impossible to have any subsequences reduce to 213.

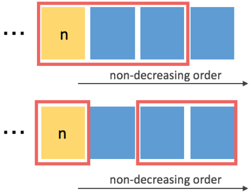


4) one n in the beginning and one at the end of p'

b) Subsequences start with n.
Same argument as 2). (two n in the beginning)

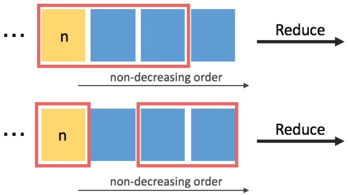
4) one n in the beginning and one at the end of p'

b) Subsequences start with n.
Same argument as 2). (two n in the beginning)



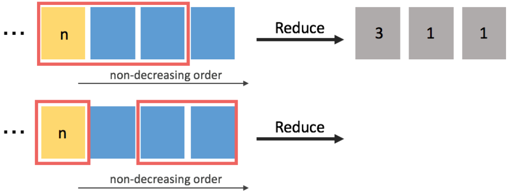
4) one n in the beginning and one at the end of p'

b) Subsequences start with n.
Same argument as 2). (two n in the beginning)



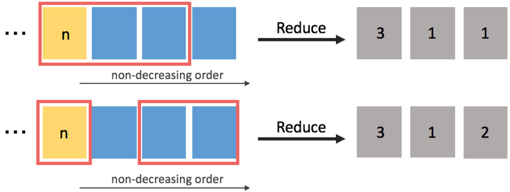
4) one n in the beginning and one at the end of p'

b) Subsequences start with n.
Same argument as 2). (two n in the beginning)



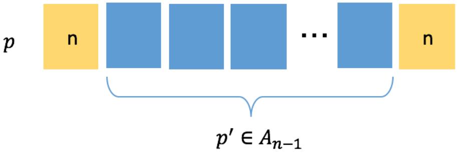
4) one n in the beginning and one at the end of p'

b) Subsequences start with n.
Same argument as 2). (two n in the beginning)



4) one n in the beginning and one at the end of p'

c) Subsequences without any n's.
Because $p' \in A_{n-1}$, no subsequence of p' reduces to a sequence from $\{132,231,213\}$.



$\rightarrow$ Create a member of A_n.

Bijection

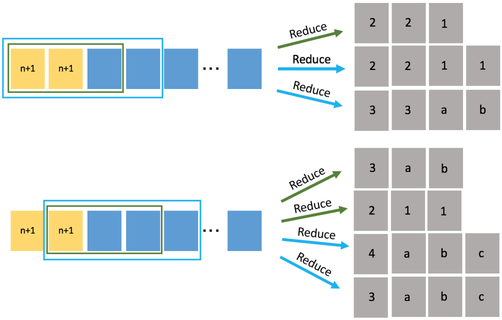
Similar to our proof of Lemma 1,

- "between" the digits of p'/q'
  $\rightarrow$ always yields a subsequence which reduces to 132 or 231.

Bijection

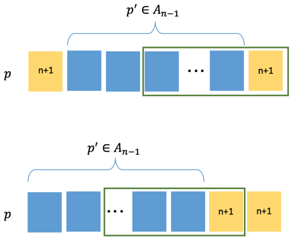
Similar to our proof of Lemma 1,

- "between" the digits of p'/q'
  $\rightarrow$ always yields a subsequence which reduces to 132 or 231.
- two $(n+1)$'s in the beginning
  $\rightarrow$ always creates $q \in S^2_{n+1}(132, 231, 2134)$.



Bijection

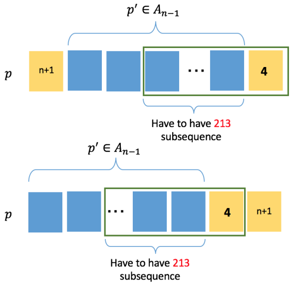
- two $(n+1)$'s at the end of p'
- one $(n+1)$ in the beginning and the other at the end of p'



- None of permutations with one n in the beginning reduces to one of $\{132, 231, 2134\}$.
- 2134 is the only permutation with biggest digit at the end

Bijection

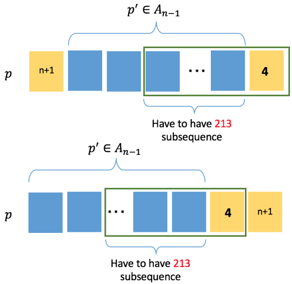
- two $(n+1)$'s at the end of p'
- one $(n+1)$ in the beginning and the other at the end of p'



- None of permutations with one n in the beginning reduces to one of $\{132,231,2134\}$.
- 2134 is the only permutation with biggest digit at the end

Bijection

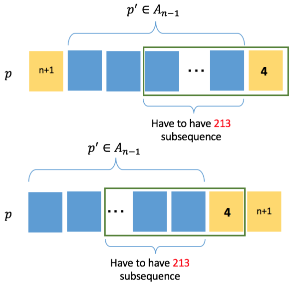
- two $(n+1)$'s at the end of p'
- one $(n+1)$ in the beginning and the other at the end of p'



- None of permutations with one n in the beginning reduces to one of $\{132,231,2134\}$.
- 2134 is the only permutation with biggest digit at the end
- $p' \in A_{n-1}$ has no subsequence of 213. $\rightarrow$ Always create a member of $S^2_{n+1}(132, 231, 2134)$.

Bijection

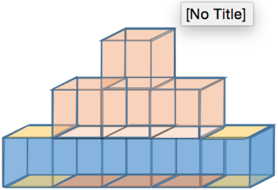
- two $(n+1)$'s at the end of p'
- one $(n+1)$ in the beginning and the other at the end of p'



- None of permutations with one n in the beginning reduces to one of $\{132,231,2134\}$.
- 2134 is the only permutation with biggest digit at the end
- $p' \in A_{n-1}$ has no subsequence of 213. $\rightarrow$ Always create a member of $S^2_{n+1}(132, 231, 2134)$.
- For these cases, $|A_{n-1}|$ is the number of permutations.

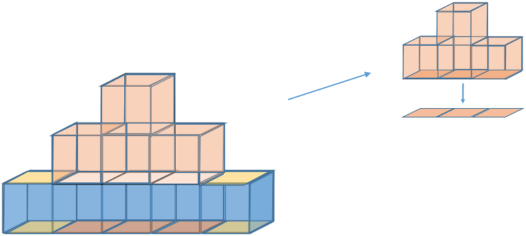
Bijection

(n+1) pile



Bijection

(n+1) pile

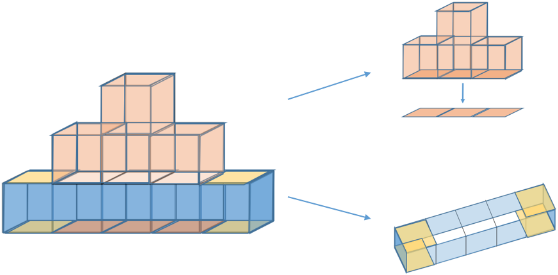


- **two (n+1)'s in the beginning**

→ Always creates a member of p

Bijection

(n+1) pile



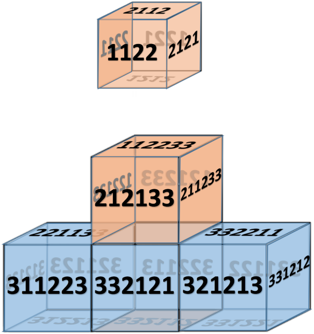
- **two (n+1)'s in the beginning**

→ Always creates a member of p

- **two (n+1)'s at the end**
- **one (n+1) in the beginning and one at the end**

→Sometimes creates a member of p

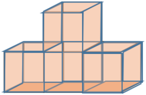
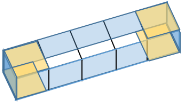
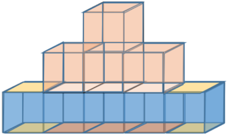
As a result, we have established a recursive bijection between the triangular piles of cubes and the member of $S^2_{n+1}(132, 231, 2134)$.
Earier, we have found that $SA(n) = 2n^2 + 6n - 2$ for $n \geq 1$. Therefore,



Theorem

$|S^2_{n+1}(132, 231, 2134)| = 2n^2 + 6n - 2 = SA(n)$

Bijection

 +  = 

SA(n-1) + 4n+4 = SA(n)

$|B_n| + 2|A_n| = |B_{n+1}|$

$|B_n| + 2(2n+2) = |B_{n+1}|$