Analyzing the Smarts Pyramid Puzzle

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Outline

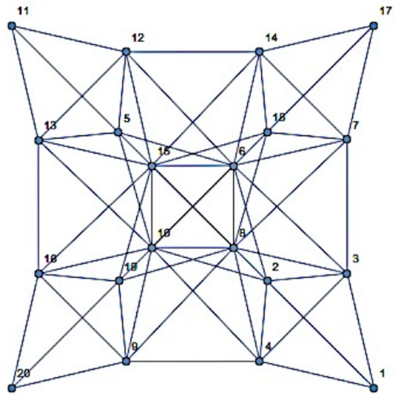
1. Defining the problem and its solutions
2. Introduce and prove Burnside's Lemma
3. Proving the results

The Smarts Pyramid

- 20 balls
- 10 pairs
- 5 colors
- No two adjacent balls share the same color

The Smarts Pyramid

- **Definition**: *A graph* $G = (V, E)$ *is an ordered pair where* V *and* E *are the set of vertices and edges of* G*, respectively.*
- **Defintion**: *A graph* $G' = (V', E')$ *is a subgraph of* $G = (V, E)$ *if* $V' \subset V$ *and* $E' \subset E$.
- **Notation**: Vertex $v = (a, b, c, d)$



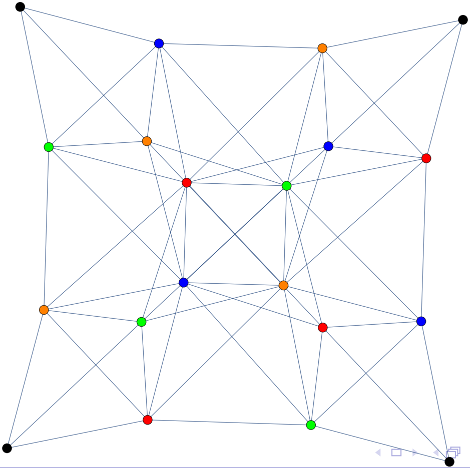
- **Definition**: *A matching M of a graph G is a subgraph of G such that the edges are pairwise disjoint from each other.*
- **Definition**: *A graph with an edge between every pair of two vertices is a complete graph.*
- **Definition**: *A complete graph with n vertices is a K_n graph.*

Matchings: T_i-matchings

- **Definition**: *A tetrahedron is a K_4 subgraph of G.*

- **Definition**: *A matching M of graph G contains a tetrahedron if M contains two edges whose corresponding vertices in G are adjacent and form a K_4 subgraph of G.*
- **Definition**: *A T_i-matching is a matching M where $i \in \mathbb{N}$ is the number of tetrahedra contained in M.*

- **Definition**: *A* *k-coloring* *of the graph P is an assignment of colors represented by integers such that no pair of adjacent vertices share a common color.*



Given the coordinates (a, b, c, d), 24 permutations can be applied to it. The 24 permutations consist of:

- Single Reflections: $(a, b, c, d) \rightarrow (b, a, c, d)$.
- Double Reflections: $(a, b, c, d) \rightarrow (b, a, d, c)$.
- Rotations: $(a, b, c, d) \rightarrow (a, d, b, c)$.
- Four-Cycles: $(a, b, c, d) \rightarrow (b, c, d, a)$.
- Identity: $(a, b, c, d) \rightarrow (a, b, c, d)$

Permutation g on element x is denoted as $g \cdot x$.

- **Definition**: *Two colorings, C and C' are equivalent if and only if there exists a permutation g such that for every vertex $v \in P$, $C(g \cdot v) = C'(v)$.*
- Two matchings $M = (V, E)$ and $M' = (V', E')$ are equivalent if and only if there exists a permutation g such that for all vertices v in V, $g \cdot v \in V'$.
- **Definition**: *A coloring C is fixed by g if and only if for every vertex v of P, $C(g \cdot v) = C(v)$.*

Burnside's Lemma

Given a Group G that acts on a set S:

- **Definition**: An orbit of x under Group G is denoted by
$$orb_G(x) = \{g \cdot x | g \in G\}$$
- **Definition**: A stabilizer of x under Group G is denoted by
$$stab_G(x) = \{g \in G | g \cdot x = x\}$$
- **Definition**: The number of fixed points of g is denoted by $fix(g)$.
$$fix(g) = |\{x \in S | g \cdot x = x\}|$$
- **The Orbit-Stabilizer Theorem:** *Given finite group G which acts on a finite set S, for each* $x \in S$,
$$|stab_G(x)| \cdot |orb_G(x)| = |G|.$$

Burnside's Lemma: *Given a finite group G that acts on a finite set S.*

$$G \times X \to X$$

The number of orbits in set S is equal to the sum over all $g \in G$ of the cardinality of the subsets of S where each subset is fixed by some g.

$$\text{number of orbits} = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

Burnside's Lemma: Proof

Step 1. $\sum_{x \in S} |stab_G(x)| = \sum_{g \in G} fix(g)$

- Consider a matrix A with rows indexed by $x \in S$ and with columns indexed by $g \in G$. A will be defined such that

$$A_{xg} \begin{cases} 1, & g \cdot x = x \\ 0, & otherwise \end{cases}$$

- Given $x \in S$, the sum across a row is $|stab_G(x)|$. And, given a $g \in G$, the sum down a column is $fix(g)$.

$\sum_{x \in S} |stab_G(x)| = \sum_{x \in S}(sum\ of\ all\ 1's\ in\ row\ x) =$ the total number of 1's in the matrix.

$\sum_{g \in G} fix(g) = \sum_{g \in G}(sum\ of\ all\ 1's\ in\ column\ g) =$ the total number of 1's in the matrix.

Thus, $\sum_{x \in S} |stab_G(x)| = \sum_{g \in G} fix(g)$, as wanted.

Step 2. Orbit Stabilizer Theorem

The Orbit-Stabilizer Theorem can be rewritten from

$$|stab_G(x)| \cdot |orb_G(x)| = |G|.$$

to

$$\frac{|stab_G(x)|}{|G|} = \frac{1}{|orb_G(x)|}$$

Step 3. Putting it together

$\sum_{x \in S} |stab_G(x)| = \sum_{g \in G} fix(g) \cdots$ from Step 1.

$$\sum_{x \in S} |stab_G(x)| = \sum_{g \in G} fix(g)$$

$$\frac{1}{|G|} \sum_{x \in S} |stab_G(x)| = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

$$\sum_{x \in S} \frac{1}{|G|} |stab_G(x)| = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

Use the Orbit Stabilizer Theorem to get:

$$\sum_{x \in S} \frac{1}{|orb_G(x)|} = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

Burnside's Lemma: Proof

$$\sum_{x \in S} \frac{1}{|orb_G(x)|} = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

Focusing on the left side of the equation: Rewrite sum as the double sum $\sum_{orbits} \sum_{x \in orbit}$.

$$\sum_{orbits} \sum_{x \in orbit} \frac{1}{|orb_G(x)|} = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

Inner Sum: Given the orbit $\{x_1, x_2, \ldots, x_t\}$, each $x \in \{x_1, x_2, \ldots, x_t\}$ has an orbit, $orb_G(x)$, of size t. Thus, we get $\sum_{i=1}^{t} \frac{1}{t} = 1$.

$$\sum_{orbits} (1) = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

Burnside's Lemma: Proof

Thus,

$$number\ \ of\ \ orbits = \frac{1}{|G|}\sum_{g\in G} fix(g)$$

Results: T_5-matchings

- Three ways to construct a matching for a tetrahedron.
- With 5 tetrahedra:

$$3^5 = 243$$

- 12 distinct T_5-matchings up to equivalence

Table: Burnside's Lemma: T_5-matchings

Action	**Number**	**Matchings Fixed**	**Total**
Single Reflection	6	3	18
Double Reflection	3	3	9
Rotation	8	0	0
4-Cycle	6	3	18
Identity	1	243	243

Burnside's Lemma:

$$number\ of\ orbits = \frac{1}{|G|} \sum_{g \in G} fix(g)$$

$$number\ of\ orbits = \frac{1}{24} \times (18 + 9 + 0 + 18 + 243)$$

$$number\ of\ orbits = \frac{288}{24} = 12$$

Results: Colorings

- $\binom{20}{4} \times \binom{16}{4} \times \binom{12}{4} \times \binom{8}{4} \times \binom{4}{4} = 305,540,235,000$ ways to assign colors to vertices
- 3778 colorings
- 183 distinct colorings up to equivalence

Table: Burnside's Lemma: Colorings

$\lvert orb_{S_4}(x)\rvert$	$\lvert stab_{S_4}(x)\rvert$	Occurrences	"$fix(g)$"
24	1	139	omitted
12	2	30	360
8	3	4	64
6	4	7	126
4	6	1	20
3	8	1	21
1	24	1	23

Burnside's Lemma:

$$number\ \ of\ \ orbits = \frac{1}{|G|}\sum_{g\in G} fix(g)$$

$$number\ \ of\ \ orbits = \frac{1}{24}\times(360+64+126+20+21+23+\mathbf{3778})$$

$$number\ \ of\ \ orbits = \frac{4392}{24} = 183$$

- **The Un-Named Theorem**: *Each coloring admits exactly 6 T_5-matchings.*
- Each of the 183 colorings have 1 to 6 non-isomorphic T_5-matchings.
- There are between 183 and 1098 unique solutions to the Smarts Pyramid using T_5-matchings.

References

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Definition: A symmetry group of degree n, denoted by S_n, is the group of all permutations g on a finite set of n elements.

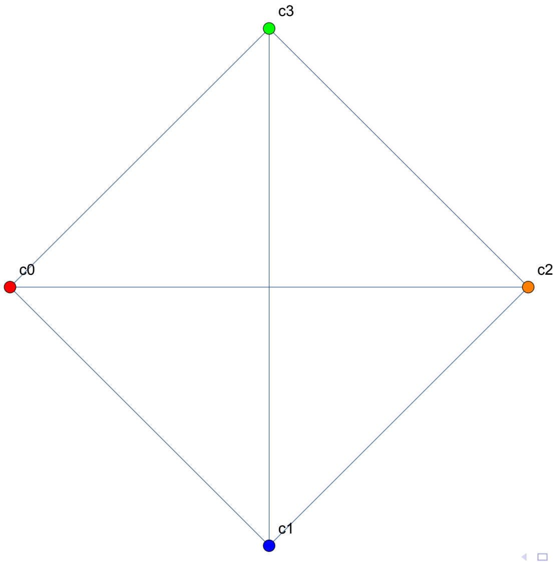
x \ y	g1	g2	g3	g4	g5	g6	g7	g8	g9	g10	g11	g12	g13	g14	g15	g16	g17	g18	g19	g20	g21	g22	g23	g24
g1	g1	g2	g3	g4	g5	g6	g7	g8	g9	g10	g11	g12	g13	g14	g15	g16	g17	g18	g19	g20	g21	g22	g23	g24
g2	g2	g1	g4	g3	g6	g5	g8	g7	g10	g9	g12	g11	g14	g13	g16	g15	g18	g17	g20	g19	g22	g21	g24	g23
g3	g3	g5	g1	g6	g2	g4	g9	g11	g7	g12	g8	g10	g15	g17	g13	g18	g14	g16	g21	g23	g19	g24	g20	g22
g4	g4	g6	g2	g5	g1	g3	g10	g12	g8	g11	g7	g9	g16	g18	g14	g17	g13	g15	g22	g24	g20	g23	g19	g21
g5	g5	g3	g6	g1	g4	g2	g11	g9	g12	g7	g10	g8	g17	g15	g18	g13	g16	g14	g23	g21	g24	g19	g22	g20
g6	g6	g4	g5	g2	g3	g1	g12	g10	g11	g8	g9	g7	g18	g16	g17	g14	g15	g13	g24	g22	g23	g20	g21	g19
g7	g7	g8	g13	g14	g19	g20	g1	g2	g15	g16	g21	g22	g3	g4	g9	g10	g23	g24	g5	g6	g11	g12	g17	g18
g8	g8	g7	g14	g13	g20	g19	g2	g1	g16	g15	g22	g21	g4	g3	g10	g9	g24	g23	g6	g5	g12	g11	g18	g17
g9	g9	g11	g15	g17	g21	g23	g3	g5	g13	g18	g19	g24	g1	g6	g7	g12	g20	g22	g2	g4	g8	g10	g14	g16
g10	g10	g12	g16	g18	g22	g24	g4	g6	g14	g17	g20	g23	g2	g5	g8	g11	g19	g21	g1	g3	g7	g9	g13	g15
g11	g11	g9	g17	g15	g23	g21	g5	g3	g18	g13	g24	g19	g6	g1	g12	g7	g22	g20	g4	g2	g10	g8	g16	g14
g12	g12	g10	g18	g16	g24	g22	g6	g4	g17	g14	g23	g20	g5	g2	g11	g8	g21	g19	g3	g1	g9	g7	g15	g13
g13	g13	g19	g7	g20	g8	g14	g15	g21	g1	g22	g2	g16	g9	g23	g3	g24	g4	g10	g11	g17	g5	g18	g6	g12
g14	g14	g20	g8	g19	g7	g13	g16	g22	g2	g21	g1	g15	g10	g24	g4	g23	g3	g9	g12	g18	g6	g17	g5	g11
g15	g15	g21	g9	g23	g11	g17	g13	g19	g3	g24	g5	g18	g7	g20	g1	g22	g6	g12	g8	g14	g2	g16	g4	g10
g16	g16	g22	g10	g24	g12	g18	g14	g20	g4	g23	g6	g17	g8	g19	g2	g21	g5	g11	g7	g13	g1	g15	g3	g9
g17	g17	g23	g11	g21	g9	g15	g18	g24	g5	g19	g3	g13	g12	g22	g6	g20	g1	g7	g10	g16	g4	g14	g2	g8
g18	g18	g24	g12	g22	g10	g16	g17	g23	g6	g20	g4	g14	g11	g21	g5	g19	g2	g8	g9	g15	g3	g13	g1	g7
g19	g19	g13	g20	g7	g14	g8	g21	g15	g22	g1	g16	g2	g23	g9	g24	g3	g10	g4	g17	g11	g18	g5	g12	g6
g20	g20	g14	g19	g8	g13	g7	g22	g16	g21	g2	g15	g1	g24	g10	g23	g4	g9	g3	g18	g12	g17	g6	g11	g5
g21	g21	g15	g23	g9	g17	g11	g19	g13	g24	g3	g18	g5	g20	g7	g22	g1	g12	g6	g14	g8	g16	g2	g10	g4
g22	g22	g16	g24	g10	g18	g12	g20	g14	g23	g4	g17	g6	g19	g8	g21	g2	g11	g5	g13	g7	g15	g1	g9	g3
g23	g23	g17	g21	g11	g15	g9	g24	g18	g19	g5	g13	g3	g22	g12	g20	g6	g7	g1	g16	g10	g14	g4	g8	g2
g24	g24	g18	g22	g12	g16	g10	g23	g17	g20	g6	g14	g4	g21	g11	g19	g5	g8	g2	g15	g9	g13	g3	g7	g1

Theorem 1

Theorem: There is only one coloring, up to equivalence, that is symmetric over all permuations.

- **Definition**: Two colors c_1 and c_2 are interchangeable if all vertices colored c_1 can be taken to vertices colored c_2 by a graph automorphism, i.e., by recoloring c_1 to c_2 and c_2 to c_1, we get an equivalent coloring.
- Assume P is in a T_5-matching.
- Without loss of generality, the center tetrahedron is colored 0, 1, 2, and 3.
- 0, 1, 2, and 3 are interchangeable.

Theorem 1



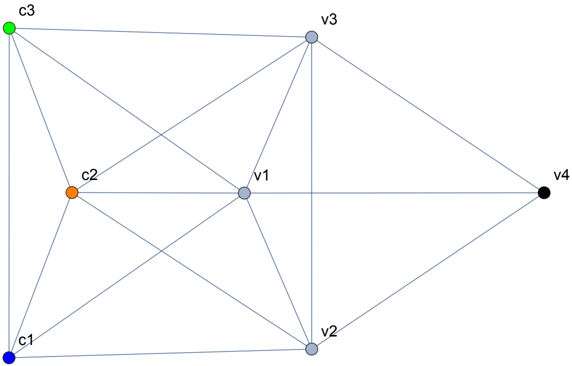
$c_0 = 0$

$c_1 = 1$

$c_2 = 2$

$c_3 = 3$

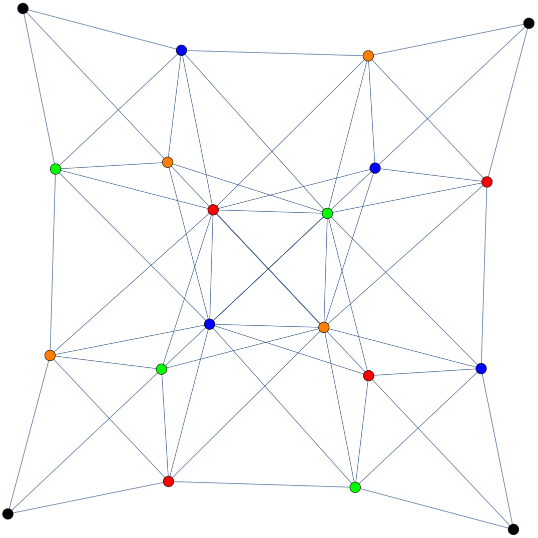
- 4 is not interchangeable with any other color, thus any symmetry takes 4 to itself.
- The four corners of the pyramid are colored 4.



Theorem 1

- Vertices v_1, v_2 and v_3 each have one option for a valid coloring.
- c_2 and v_4 are adjacent to v_1, v_2, and v_3, thus v_1, v_2, and v_3 may not be colored 2 or 4. They may also not share a color.
- v_1 is adjacent to c_1, c_2, and c_3, thus the only valid color for v_1 is 0.
- v_3 is adjacent to c_3, which is colored 3. So, v_3 must be colored 1.
- v_2 is colored 3

Theorem 1



Theorem: Any coloring of the Smarts Pyramid admits exactly six T_5-matchings.

1. Any coloring of the pyramid has at most six T_5-matchings.

2. Any coloring of the pyramid has at least six T_5-matchings.

Theorem 2: Part 1

Step 1. $\forall i \neq j$, the colorings of T_i and T_j do not contain exactly the same colors.

- Given four of each color $(1, 2, 3, 4,$ and $5)$, we have
  $$q = [1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5]$$
  where q is a list representing available color assignments.
- Assume towards contradiction that there exists T_1 and T_2 such that their colorings both contain the colors 1, 2, 3, and 4.
- The remaining available color assignments are then
  $$q = [1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 5, 5].$$
- By the pigeonhole principle, at least one tetrahedron will have two 5's in its coloring.
- Thus, $\forall i \neq j$, T_i and T_j have different colorings.

Step 2. There is only one way to color five tetrahedra such that no two tetrahedra have the same coloring.

- A tetrahedron is made up of four different colors.
- With 5 colors to choose from, the number of ways to color a tetrahedron is $\binom{5}{4} = 5$.
- Because of we must color 5 tetrahedra, there is only one way to uniquely color 5 tetrahedra.

Step 3. Given the colorings in Step 2, there are six T_5-matchings compatible with it.

- **Notation**: An edge whose vertices are colored will be denoted by $\underline{[a, b]}$, where a and b represents the vertices' colors. The union of two edges is a matching of a tetrahedron.
- Any given edge $[a, b]$ can be paired with three other edges to make a matching of a colored tetrahedron.
- Without loss of generality, if [1,2] is paired with the edge [3,4], then the edges that contain a vertex colored 5, or [*,5], are [1,5], [2,5], [3,5], and [4,5]. The other remaining edges are: [1,3], [1,4], [2,3], and [2,4].
- No edge [*, 5] can be paired with another [*,5]. So, each [*,5] is paired with an edge that does not contain a vertex which is colored 5, otherwise, by the pigeonhole principle, a [*,5] will be paired with another [*,5].
- Each edge without a five can be paired with exactly two [*,5].

Theorem 2: Part 1

- Each edge without a five can be paired with exactly two [*,5].
- Edge [1,3] can be paired with either [2,5] or [4,5]. If [1,3] is paired with [2,5], then [1,4] must be paired with [3,5]
- Then [2,4] must be paired with [1,5], and [2,3] with [4,5].
- Likewise, if [1,3] is paired with [4,5] instead of [2,5], then the remaining matchings must be:

$$[2,3] \text{ to } [1,5]$$
$$[2,4] \text{ to } [3,5]$$
$$[1,4] \text{ to } [2,5]$$

The first tetrahedron matched has three matchings to choose from. The remaining eight edges can be paired in two different ways. So, there are at most six matchings for five tetrahedra that each have a different coloring.

Theorem 2: Part 2

- From the set of five colored tetrahedra of P, choose one tetrahedron, let's say T_1 that has no matching assigned. This tetrahedron's coloring, without loss of generality, includes the colors 1, 2, 3, and 4.
- This means that the coloring of the other four tetrahedra include the colors as follows:

The coloring of T_2 includes the colors 1, 2, 3, and 5

The coloring of T_3 includes the colors 1, 2, 4, and 5

The coloring of T_4 includes the colors 1, 3, 4, and 5

The coloring of T_5 includes the colors 2, 3, 4, and 5

Theorem 2: Part 2

- In the matching of T_1, the vertex colored 1 can be in an edge with the vertex colored 2, 3, or 4.
- Assume that the matching of T_1 is made of of edges [1, 2] and [3, 4].
- T_2 and T_3 also contain the colors 1 and 2. T_2 also contains 3 and 5. T_3 also contains 4 and 5.
- T_1, T_2, and T_3 are equivalent up to recoloring.
- T_1 contains [3,4], so T_4 must have [3,5] or [4,5]
- If T_4 has [3,5], then T_5 has [4,5] and if T_4 has [4,5], then T_5 has [3,5].
- From the three choices for matching T_1 and the two choices for matching T_4 and T_5, we conclude that there are at least six ways to match the five tetrahedra.