The Nine Card Problem: Combinatorics and Magic Tricks

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Outline

What is the Nine Card Problem?

1. Take any nine cards from a deck. Put them into a stack and look at the third card from the top.

2. Remember the rank and suit of the third card.

3. Now, spell out the rank. Every time you state a letter, put the top card down on the table.

4. Once you have spelled out the word, pick up the stack and put it in the bottom of the pile in your hand.

5. After spelling the rank, spell out "of," then the suit.

6. After spelling the suit, spell "MAGIC."

7. Flip over the last card. It should be the card you spelled out.

Definitions

Deck: the set of D cards that are picked. D will be known as the Size.

Step: One round of spelling a word, placing cards on the table, and picking them up.

Dealt Card: A card placed on the table.

Shuffle: Picking up cards once they have been dealt.

Shuffle Length, L_i: The number of cards dealt before picking them up at the end of a step i.

What the Shuffle Does

Each Shuffle places the dealt cards in reverse order on the bottom.

Example

Let $x = \{123456789\}$. The function $iDeal(x)$ takes a set x and returns a shuffle with length i.
If $i = 2$, then $2Deal(x) = 12|3456789 = 3456789\textcolor{red}{21}$
If $i = 3$, then $3Deal(x) = 123|456789 = 456789\textcolor{red}{321}$
If $i = 4$, then $4Deal(x) = 1234|56789 = 56789\textcolor{red}{4321}$

Groups for all steps

Note that some words, like "Ace" and "Two," have the same amount of deals.

Rank

- 3 cards dealt: (ace, two, six, ten)
- 4 cards dealt: (four, five, nine, jack, king)
- 5 cards dealt: (three, seven, eight, queen)

"of"

- 2 cards dealt: (of)

Suit

- 5 cards dealt: (clubs)
- 6 cards dealt: (spades, hearts)
- 8 cards dealt: (diamonds)

The Exhaustive proof of the 9 card trick

In order to find the total amount of possible deals, multiply the number of groups in each set of Rank, "Of", and Suit.

There are $3 * 1 * 3 = 9$ sets associated with each group permutation.

Finally, all permutations will be subjected to $5Deal(x)$ [Or "Magic"] so if the 3rd card is in the 5th position by the penultimate step, then we have proved it for all cases.

All Possible Combinations of Shuffles

		Unit	→ Step 1	→ Step 2	→ Step 3
3,2,5	:	123|456789	→ 45|6789321	→ 67893|2154	→ 215439876
3,2,6	:	123|456789	→ 45|6789321	→ 678932|154	→ 154239876
3,2,8	:	123|456789	→ 45|6789321	→ 67893215|4	→ 451239876
4,2,5	:	1234|56789	→ 56|7894321	→ 78943|2165	→ 216534987
4,2,6	:	1234|56789	→ 56|7894321	→ 789432|165	→ 165234987
4,2,8	:	1234|56789	→ 56|7894321	→ 78943216|5	→ 561234987
5,2,5	:	12345|6789	→ 67|8954321	→ 89543|2176	→ 217634598
5,2,6	:	12345|6789	→ 67|8954321	→ 895432|176	→ 176234598
5,2,8	:	12345|6789	→ 67|8954321	→ 89543217|6	→ 671234598

Pseudo-Fixed Point

Definition

A Pseudo-Fixed Point (PFP) is a card, f_i, in the original distribution $S = 1, 2, ..., f_i, ..., D$ such that given three steps, the card remains in the same position relative to its counterparts with different shuffle lengths.

Equivalence Classes

Definition

An Equivalence Class for j is a group of shuffles $\{L_i\}$ such that a card will be placed on the table.

Example

Consider our earlier example.
$2Deal(x) = 12|3456789 = 345678921$
$3Deal(x) = 123|456789 = 456789321$
$4Deal(x) = 1234|56789 = 567894321$
Here, the equivalence class for $j = 2$ would be $\{2, 3, 4\}$ as they all move the second card.

Insight of equivalence classes

The trick uses a sleight of hand method that makes the viewer think that the card is moving.

If a magician wants a certain card to move, they will have to deal at least the smallest shuffle in the equivalence class.

The General Proof for the 9 card Problem: Rank Step

Proposition: For the 9 card problem, 3 is the only PFP.

Proof.
First, start with the original placement of the cards.

$$123456789$$

The first shuffle will either be $3deal(x), 4deal(x)$,or $5deal(x)$. Note that each function will deal the first three cards.
The first shuffle returns:

$$123456789 \to a_1 a_2 a_3 a_4 a_5 a_6 321$$

Where a_i are arbitrary cards that depend on the shuffle size. □

The General Proof for the 9 card Problem: Of Step

Proof.

The second permutation follows from the first like so:

$$a_1 a_2 a_3 a_4 a_5 a_6 321 \to a_3 a_4 a_5 a_6 321 a_2 a_1$$

The first three cards simply move up two spaces. □

The General Proof for the 9 card Problem: Suit Step

Proof.

The third permutation follows from the second. Similar to the first situation, our three functions will either be $5deal(x), 6deal(x)$,or $8deal(x)$. All of them will at least move the 3rd card. Therefore the final placement will be:

$$a_3 a_4 a_5 a_6 321 a_2 a_1 \rightarrow a_1 a_2 a_3 a_4 3 a_5 a_6 a_7 a_8$$

Therefore, $f = 3$ is a pseudo-fixed point and every possible permutation of different shuffles will lead to 3 being in the fifth spot. □

Generalizing the Nine Card Problem

The 9 card problem can be generalized to include differing deck sizes, D, as well as different variable shuffle lengths L_i.

In order to show a variation of the problem, there needs to be one additional definition.

New Definition of l_i

Before continuing on, it is important to note here that changing the shuffle lengths will require bounds on the smallest possible length for a step and the total sum of those.

Definition

The smallest length in a step l_i is the minimum in an equivalence class of a step.

The sum of these minimums need to be bounded above and below to guarantee that a PFP exists.
In our example, $l_1 = 3$, $l_2 = 2$, and $l_3 = 5$.

PFPs in a generalized proof

1a. $3(D-1) \geq l_1 + l_2 + l_3 \geq D + 1$ must occur for a PFP

1b. So in our original case, $3 + 2 + 5 = 11 \geq 10$

2a. $(l_1 + l_2 + l_3) - D = s$, where $s \geq 1$

2b. $(3 + 2 + 5) - 10 = 1$

3a. Depending on the minimum of l_1, l_3, and s, the number of PFPs and their final positions can be determined.

3b. The minimum of 3, 5, and 1 is 1, or s.

PFPs and their final positions

Table : PFP and final positions

	$\min\{l_1, l_3, s\}$	Additional Restriction	PFPs	Final Positions
1	l_1	$l_2 \leq D - l_1$	$1, 2, ..., l_1$	$l_2 + 1, l_2 + 2, ..., l_2 + l_1$
2	l_1	$l_2 > D - l_1$	$1, 2, ..., l_1$	$l_2 + 1, l_2 + 2, ..., D, l_2, l_2 - 1, ...D - l_1 + 1$
3	l_3	$l_3 > D - l_2$	$1, 2, ..., l_3$	$l_2 + 1, l_2 + 2, ..., D, l_2, l_2 - 1, ...D - l_3 + 1$
4	l_3	$l_3 \leq D - l_2$	$l_1 - s + 1, l_1 - s + 2, ..., D - l_2$	$D - l_3 + 1, ..., D - 1, D$
5	s	N/A	$l_1 - s + 1, ..., l_1$	$l_1 + l_2 - s + 1, ..., l_1 + L_2$

An Example for Case 5

Suppose the order of the first and the last shuffle in the Nine Card Problem were switched, so it was "Suit of Rank".
Here, $l_1 = 5$, $l_2 = 2$, $l_3 = 3$, and $s = 1$.
Therefore, s would be the minimum.
According to the table, the PFP would be the
$l_1 - s + 1 = 5 - 1 + 1 =$5th card.
It's final position would be $l_1 + l_2 - s + 1 = 5 + 2 - 1 + 1 =$ 7th in the deck.

Example for Case 5 Test

Because of equivalence classes, all that needs to be done is to check the minimum lengths and see if they work.
Consider the following example with [5, 2, 3, or Clubs of Ace] and deck size $D = 9$:

Original Order 123456789

Step 1: $12345|6789 \to 678954321$

Step 2: $67|8954321 \to 895432176$

Step 3: $895|432176 \to 432176598$

Conclusion

1: Learned of the 9 Card Problem.

2: Saw why and how the 9 Card Problem worked.

3: Changed the 9 Card Problem slightly and saw it still works.

Bibliography

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