Mathematical Modeling

Math 396 Fall 2008 | Fowler 110 Thu 1:30- 2:55pm

 | http://faculty.oxy.edu/ron/math/396/08/

Class 6: **Thursday October 2**

TITLE Discrete Probabilistic Modelling: Markov Chains

CURRENT READING Meerschaert, Chapter 8 and Giordano, Chapter 6

SUMMARY

This week we will be introduced to discrete stochastic modelling.

DEFINITION: **Markov Chain**

A **Markov chain** is a discrete-time stochastic model.In other words, it is a sequence of random changes in a state variable X_n where the probability of "jumping" from state X_n to state X_{n+1} depends only on the state X_n.

Generally, we let $X_n \in \{1, 23, 4, 5, \ldots, m\}$. Given a sequence of numbers $\{X_n\}$ that is a Markov Chain we define the (i, j) entries of a probability transition matrix P

$$p_{ij} = \Pr(X_{n+1} = j | X_n = i)$$

In this case, given the entries in the probability transition atrix, all future values of the variable X_n can be determined (probabilistically), once one starts with X_0.

EXAMPLE 8.2 (Meerschaert)

Describe the behavior of tyhe following Markov chain. The state variable $X_n \in \{1, 2, 3\}$ If $X_n = 1$, then $X_{n+1} = 1, 2 or 3$ with equal probability.
If $X_n = 2$ then $X_{n=1} = 1$ with probability 0.7, and $X_{n+1} = 2$ with probability 0.3.
If $X_n = 3$ then $X_{n+1} = 1$ with probaility 1.

State Transition Diagram

A diagram which shows the behavior of the state variable can br drawn below

Probability Transition Matrix

We can also place the probabilities of each transition into matrix form.

Let's calculate the probabilities of the X_2 achieving each possible value.

$\Pr(X_2 = 1) =$

$\Pr(X_2 = 2) =$

$\Pr(X_2 = 3) =$

GROUPWORK

Also calculate $\Pr(X_3 = 1)$, $\Pr(X_3 = 2)$ and $\Pr(X_3 = 3)$

What we'd really like is an expression for $\pi_n(i) = \Pr(X_n = i)$.

If we let $\vec{\pi}_n$ be a vector with the components of $\pi_n(1)$, $\pi_n(2)$ and $\pi_n(3)$ then we can write $\vec{pi}_{n+1} = \vec{\pi}_n P$.

Steady State Distribution

What we'd like to do is find out if $\lim_{n\to\infty} \vec{\pi}_n$ exists and is a finite vector, if it does it is called the steady state distribution of the Markov chain.

$\lim_{n\to\infty} \vec{\pi}_n =$

Homework Question for Math 396

Suppose we have a simple phone system, where X_t is the state of the phone at time t. $X_t = 0$ when phone is free, and $X_t = 1$ when phone is busy.

$P_t(i, j)$= probability that the phone goes from state i to state j during a time interval. (We assume only one phone call per time interval.)

$p = 1/4$ = probability phone call comes in during the time interval
$q = 1/6$ = probability phone call is completed in the time interval

1) Determine the probability transition matrix, P for this system.
$P(0, 1) =$
$P(1, 0) =$
$P(0, 0) =$
$P(1, 1) =$

We can write P as a matrix, where $P_{1,1} = P(0, 0)$, $P_{1,2} = P(0, 1)$, $P_{2,1} = P(1, 0)$ and $P_{2,2} = P(1, 1)$

2) Draw the state transition diagram

The probability that if the phone is free at time 0, it is free at time $t = 2$ is $P_2(0, 0)$
$P_2(0, 0) = P_0(0, 0)P_1(0, 0) + P_0(0, 1)P_1(1, 0) =$
$P_2(0, 0) = P^2_{1,1}$
It turns out that any time t that P^t gives you the probability transition matrix of the phone system at state t.

3) What is the long range behavior of the phone system? Compute $\lim_{t\to\infty} P^t$.
Compute P^∞. How? What do we need to do?