Mathematical Modeling

Week 2: **Thursday September 4**

TITLE Modeling Using Proportionality
CURRENT READING Giordano, 2.2

SUMMARY
Let's start our reviews of various types of modelling by reviewing models that involve proportionality and model fitting (using least squares).

DEFINITION: **proportionality**

Two variables y and x are **proportional** (to each other) if one is always a constant multiple of the other." (Giordano, Weir & Fox, page 2.) We denote this relationship as $y \propto x$ if and only if $y = kx$ where k is a non-zero constant.

Consider the following table of "famous proportionalities."

Hooke's Law: $F = kS$, where F is the restoring force in a spring stretched or compressed a distance S.
Newton's Law: $F = ma$, where a is the acceleration of a mass m subject to an external force F.
Ohm's Law: $V = iR$ where i is the current induced by a voltage V across a resistance R.
Boyle's law: $V = \frac{k}{p}$, where under a constant temperature, the volume V of an ideal gas is proportional to the pressure p.
Einstein's theory of relativity: $E = mc^2$, where the energy E equivalent of a mass m is proportional with constant c^2, the speed of light squared.
Kepler's third law: $T = cR^{3/2}$, where T is the period (in days) and R is the mean distance to the sun.

Problem
Consider the data in `S:\Math Courses\Math396\Kepler.xls`. Try to obtain a value for the constant of proportionality c in Kelper's Third Law. (What are your units?)

What are some assumptions you have to make? Explain what you have to do to obtain your answer for c.

Approximation Theory

In approximation theory we have a set of m data points (x_k, y_k) for which we do not know what the actual function $f(x)$ which reflects the relationship between the input variable x and the output y.

Suppose we define the **deviation** as $\delta_k = P(x_k) - y_k$ and find a function $P(x)$ such that the total deviations between the function $P(x)$ and the data points (x_k, y_k) is minimised. There is more than one way to do this.

We define a function E which represents the total deviation we are trying to minimize and we want to find P which minimizes E, where E can have different forms.

Some Ways To Formulate E Are:

$$E_1 = \sum_{k=1}^{m} |P(x_k) - y_k|$$

OR

$$E_\infty = \max_{1 \le k \le m} |P(x_k) - y_k|$$

OR

$$E_2 = \sum_{k=1}^{m} [P(x_k) - y_k]^2$$

From statistics we know that if the data are **normally distributed** then the square error (E_2) is the best form of the error to use to measure how well $P(x)$ is approximating the unknown function $f(x)$ represented by the data y_k.

Linear Fit

If we assume that the polynomial we choose for $P(x)$ is linear so that $P(x) = ax + b$ then the problem of finding P becomes a minimization problem. If we consider E is a function of the parameters a and b what it the problem we have to solve?

Therefore

$$a = \frac{\overline{xy} - \overline{x} \cdot \overline{y}}{\overline{x^2} - \overline{x}^2} \qquad b = \frac{\overline{x^2} \cdot \overline{y} - \overline{xy} \cdot \overline{x}}{\overline{x^2} - \overline{x}^2}$$

The line $P(x) = ax + b$ is known as the "least squares" line, or "line of best fit" or "regression line"

Example

Consider the following data from Table 3.3 of Giordano, Fox & Weir.

x_i	y_i	x_i^2	$x_i y_i$	$P(x_i)$	$\lvert y_i - P(x_i)\rvert$
0.5	0.7				
1.0	3.4				
1.5	7.2				
2.0	12.4				
2.5	20.1				

Problem

Attempt to predict the value of the variable y when $x = 2.25$ assuming a relationship of $y = Ax^2$.

Fitting Data to Nonlinear Functions: OR Making Nonlinear Relationships Appear Linear

Isn't there some way we could transform the equation $y = be^{ax}$ and $y = bx^a$ so that a linear relationship would appear? Then we could use our previously defined normal equations of least-squares fit for a line.

Think about introducing some new variables Y and X such that there is a linear relationship between Y and X even though y and x are non-linearly related.

A Harder One

How could you pick Y and X so that you could solve the normal equations and fit data to $y = \alpha x e^{\beta x}$?