Numerical Analysis

Math 370 Spring 2009 — MWF 11:30am - 12:25pm Fowler 110

 — `http://faculty.oxy.edu/ron/math/370/09/`

Worksheet 20

SUMMARY Application of Interpolation: Numerical Differentiation
READING Burden & Faires, 167–180; Mathews & Fink Section 6.1

Approximating Derivatives
We shall be using our knowledge of Lagrange Interpolation to come up with formulas that allow us to approximate the **derivative** of an unknown function at any point, even though we are only given a number of values of the function at specific nodes.

From Calculus we know that if we know the function at $f(x_0)$ and $f(x_0 + h)$ then an approximation to the derivative of $f(x)$ at x_0, $f'(x_0)$ can be written as:

EXAMPLE

1. Consider the function $f(x) = \ln(x)$ at $x = 2$. In small groups of 2 or 3, obtain an approximation of the derivative, $f'(x)$, at $x = 2$ using different values of h less than 0.5. Also compute the actual error your approximation makes.

2. Do you see any relation between the choice of h and the size of the error?

3. What can we do if we want to get a bound on how large the actual error could get?

4. Write down the Lagrange interpolating polynomial $P(x)$ for a function $f(x)$ given that you know the function at $(x_0, f(x_0))$ and $(x_1, f(x_1))$ (where $x_1 = x_0 + h$). HINT: what is the degree of the Lagrange Interpolating polynomial (how many nodes do we have?)?

5. Write down the error term $e(x)$:

RECALL

The error between a function $f(x)$ and its Lagrange interpolating polynomial $P(x)$ derived from the nodes $x_0, x_1, \ldots x_n$ on [a,b] is given by

$$e(x) = |f(x) - P(x)| = \left|\frac{f^{(n+1)}(\xi(x))}{(n+1)!}\right| |(x - x_0)(x - x_1) \cdots (x - x_n)|$$

where $\xi(x)$ is in (a, b)

6. Let's write down an exact expression for $f(x)$ in terms of $e(x)$ and $P(x)$

7. and differentiate this exact expression (with respect to x:

8. Now, evaluate this expression for $f'(x)$ at $x = x_0$

9. and use the fact that $x_1 = x_0 + h$ to simplify this expression. Look familiar?

10. Suppose that the maximum value that f' has on $[x_0, x_0 + h]$ is M, then we can write down an error bound for our approximation to $f'(x_0)$

EXAMPLE

Compute theoretical error bounds for the approximations to $f'(2)$ that you had previously made. Compare the actual error to theoretical error bound. What do you see?

This particular kind of approximation for $f'(x_0)$ is called a **Forward Difference Formula**. If you replace h with $-h$ then you get the

Backward Difference Formula

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}$$

(n+1)-point Difference Formulae
In general, if one has $n+1$ data points for $f(x)$ at $x_0, x_1, \ldots x_n$ using Lagrange interpolation we can express the function as:

$$f(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

Differentiating produces...

$$\begin{aligned} f'(x) &= \sum_{k=0}^{n} f(x_k)L'_{n,k}(x) + \frac{d}{dx}\left[\frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!}\right] f^{(n+1)}(\xi(x)) \\ &\quad + \frac{(x-x_0)(x-x_1)\ldots(x-x_n)}{(n+1)!}\frac{d}{dx}[f^{(n+1)}(\xi(x))] \end{aligned}$$

But if this evaluated at x_k then write down an expression for $f'(x_k)$ below:

Taylor Expansions
Another way to compute difference formula which approximate derivatives at a point is to use Taylor Approximations. This is useful because again one gets a sense of how good the approximation is as one is making it.
For example, let's write down the **third-order Taylor expansion** (i.e. up to $\mathcal{O}(h^3)$ about x_0 (include the error term) for the following expressions:
$f(x_0 + h) =$

$f(x_0 - h) =$

Then substract these two terms and solve for $f'(x_0)$

You can use this technique to come up with almost any standard finite difference approximation to a derivative.