Numerical Analysis

Math 370 Spring 2009 | MWF 11:30am - 12:25pm Fowler 110

 | http://faculty.oxy.edu/ron/math/370/09/

Worksheet 14

SUMMARY Polynomial Interpolation and Extrapolation
READING Recktenwald, pp. 521–538; Mathews & Fink, pp. 199–217

Interpolation

Suppose one has a set of $N+1$ tabulated data points (x_i, y_i) (i goes from 0 to N) that one believes have been obtained very accurately.
What you would like to do is obtain a functional representation which allows one to find a function $f(x)$ which agrees with the tabulated points (often called **nodes**), but is also able to give reasonable values at non-tabulated points.

DEFINITION : extrapolation and interpolation

When the proposed x value falls in the range $x_0 < x < x_N$ the required value $f(x)$ is called an **interpolated value**.
When the proposed x value falls in the range $x < x_0$ or $x > x_N$ the required value $f(x)$ is called an **extrapolated value**.

EXAMPLE

Consider the following application:
Suppose we have census data for the United States taken every ten years (in thousands).

YEAR	1940	1950	1960	1970	1980	1990	2000
POPULATION	132,165	151,326	179,323	203,302	226,542	249,633	281,421

Exercise

1. Suppose we want to find an estimate of the population of the USA in 1994 (the year I started teaching at Oxy!), how would we do it?

2. Estimate the population of the United States in 2009.

3. What are the assumptions behind this calculation?

Weierstrass Approximation Theorem
If f is defined and continuous on [a,b] and $\epsilon > 0$ is given, then there exists a polynomial $P(x)$, defined on [a,b] such that $|f(x) - P(x)| < \epsilon, \qquad$ for all $x \in [a, b]$.

Think! Pair. Share.
3. What does this mean? Write out what this theorem means, in your own words. DRAW A PICTURE! What is the usefulness of this theorem?
Then share your meaning with your nearest neighbor.

EXAMPLE

4. Recall the previous example of linear interpolation that we went through with U.S. Census Data. Consider the more general problem of having two data points (or **nodes**) (x_0, y_0) and (x_1, y_1).
Write down a formula for the value of the interpolated value $P(x)$ below.
(Do we have any restrictions on the x that P is evaluated at?)

Lagrange Polynomials
The *general* form of the interpolating polynomial $P(x)$ of degree 1 through the points (x_0, y_0) and (x_1, y_1) has the form:

$$P(x) = \frac{(x - x_1)}{x_0 - x_1} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

This can also be written as $P(x) = y_0 L_0(x) + y_1 L_1(x)$, where $L_k(x)$ are called **Lagrange Polynomials** of degree 1 for x_k. To be precise we should write these Lagrange polynomials as $L_{1,0}$ and $L_{1,1}$ to show that they are Lagrange Polynomials of degree 1 for x_0 and x_1, respectively.

So, $L_{5,2}(x)$ would be ____

Exercise

5. Let's show that the answer you wrote down to (4.) is the same thing as the form when written using Lagrange Polynomials

EXAMPLE

6. Evaluate the generic $P(x)$ linear interpolant at x_0 and x_1. What values do you expect? Does this happen?

7. What does this tell you about properties of $L_k(x_j)$? In other words, what happens when you evaluate a Lagrange Polynomial at a node? (Is there a difference whether $k = j$ or $k \neq j$)?

8. Draw a picture of $L_{1,0}(x)$ and $L_{1,1}(x)$ on the same axes below.

In general **Lagrange polynomials of degree** n, are given by

$$L_{n,k}(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

Using a Lagrange Polynomial of degree n to interpolate a function value $P(x)$ between $n+1$ function values y_k produces

$$P(x) = \sum_{k=1}^{n} L_{n,k}(x) y_k$$

Error Formula for Lagrange Interpolation

The error between a function $f(x)$ and its interpolating polynomial $P(x)$ derived from the nodes $x_0, x_1, \ldots x_n$ on [a,b] is given by

$$e(x) = |f(x) - P(x)| = \left| \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \right| |(x - x_0)(x - x_1) \cdots (x - x_n)|$$

where $\xi(x)$ is in (a, b)

GROUPWORK

Determine the second degree interpolating polynomial for the function $f(x) = 1/x$ which uses the nodes $x_0 = 2, x_1 = 2.5$ and $x_2 = 4$ to interpolate the value of the function at $x = 3$.

In other words, suppose we know that our data at the nodes comes from the function $f(x) = 1/x$, use this information to generate an interpolant for $f(x)$ and then evaluate your interpolant at $x = 3$.

(a) First write down the ______ Lagrange polynomials of degree 2

(b) Combine your answer above to form $P(x)$

(c) Use your interpolant to approximate $P(3)$

(d) How close is this to the actual value of $f(3)$? (What is the absolute error made by the interpolant?)