Numerical Analysis

Math 370 Spring 2009 MWF 11:30am - 12:25pm Fowler 110

 http://faculty.oxy.edu/ron/math/370/09/

Worksheet 13

SUMMARY Formulas for Iterative Techniques of Solving Linear Systems
READING Recktenwald, Sec 8.5, pp. 427–445; Sec. 7.1.2 and Sec 7.2.4; Mathews & Fink Section 3.6, 156–166

Consider the system

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15 \end{aligned}$$

We can re-write these equations as

$$x^{(k+1)} = \frac{7 + y^{(k)} - z^{(k)}}{4}, \qquad y^{(k+1)} = \frac{21 + 4x^{(k)} + z^{(k)}}{8}, \qquad z^{(k+1)} = \frac{15 + 2x^{(k)} - y^{(k)}}{5}$$

OR

$$x^{(k+1)} = \frac{7 + y^{(k)} - z^{(k)}}{4}, \qquad y^{(k+1)} = \frac{21 + 4x^{(k+1)} + z^{(k)}}{8}, \qquad z^{(k+1)} = \frac{15 + 2x^{(k+1)} - y^{(k+1)}}{5}$$

Which of these schemes represents Gauss-Seidel Iteration and which represents Jacobi Iteration?

Can you generalize these iterative schemes if the linear system looks like:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

Exercise

Write down the general iterative formula for Jacobi Iteration on a 3x3 system here:

Write down the general iterative formula for Gauss-Seidel Iteration on a 3x3 system here

Matrix representation of iterative schemes for linear systems

We have written down the iterative scheme implementation of Jacobi and Gauss-Seidel iteration but the more useful way to think about these schemes is using the matrix representation of the generic iterative scheme

$$\underline{x}^{(k+1)} = T\underline{x}^{(k)} + \underline{c}$$

and we'll derive how T depends on A and $\vec{c}$ depends on A and $\vec{b}$ for each method.

We will write the matrix A as the sum of three matrices D (diagonal matrix), L (lower triangular) and U (upper triangular) such that

$$A = D - L - U$$

$\boxed{\text{EXAMPLE}}$

Write down D, L and U for the original linear system on page 1

The system $A\underline{x} = \underline{b}$ can be written as

$$\begin{aligned}
(D-L-U)\underline{x} &= \underline{b} \\
D\underline{x} &= L\underline{x} + U\underline{x} + \underline{b} \\
\underline{x} &= D^{-1}(L+U)\underline{x} + D^{-1}\underline{b} \\
\underline{x}^{(k+1)} &= D^{-1}(L+U)\underline{x}^{(k)} + D^{-1}\underline{b}
\end{aligned}$$

Another choice is

$$\begin{aligned}
(D-L-U)\underline{x} &= \underline{b} \\
(D-L)\underline{x} &= U\underline{x} + \underline{b} \\
\underline{x} &= (D-L)^{-1}U\underline{x} + (D-L)^{-1}\underline{b} \\
\underline{x}^{(k+1)} &= (D-L)^{-1}U\underline{x}^{(k)} + (D-L)^{-1}\underline{b}
\end{aligned}$$

Which of the above schemes represents Jacobi Iteration and which represents Gauss-Seidel? How can you tell?

Rates of Convergence of iterative schemes for linear systems
We have written down the matrix implementation of Jacobi and Gauss-Seidel iteration in the form

$$\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$$

and derived how T depends on A and $\vec{c}$ depends on A and $\vec{b}$ for each method.

Gauss-Seidel Iteration

$$\vec{x}_{k+1} = (D - L)^{-1} U \vec{x}_k + (D - L)^{-1}\vec{b}$$

Jacobi Iteration

$$\vec{x}_{k+1} = D^{-1}(L + U)\vec{x}_k + D^{-1}\vec{b}$$

Successive Over-Relaxation (SOR)

$$\vec{x}_{k+1} = (D - \omega L)^{-1}[\omega U + (1 - \omega)D]\vec{x}_k + (D - \omega L)^{-1}\vec{b}$$

Gauss-Seidel ends up being a special case of successive over-relaxation with $\omega = 1$.

Spectral Radius
The spectral radius $\rho(A)$ of a N x N matrix A is defined as $\rho(A) = max|\lambda|$, where λ is an eigenvalue of A.

Properties of the Spectral Radius
(a) $||A||_2 = \sqrt{\rho(A^T A)}$
(b) $\rho(A) \leq ||A||$, for any "natural matrix norm" (i.e. a norm which also applies to vectors)

The importance of the spectral radius of a matrix is that it allows us to say a lot about the convergence and rate of convergence of iterative schemes of the form $\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$

$\boxed{\text{bf THEOREM}}$

The iterative scheme $\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$ generates a sequence $\{\vec{x}_n\}$ which converges to the unique solution of $\vec{x} = T\vec{x} + \vec{c}$ for any initial guess $\vec{x}_0$ if and only if $\rho(T) < 1$.

COROLLARY
If $||T|| < 1$ for any natural matrix norm and c is a given vector then the iterative scheme $\vec{x}_{k+1} = T\vec{x}_k + \vec{c}$ converges to $\vec{x}$ and the following error bound holds:

$$||\vec{x} - \vec{x}_k|| \leq ||T||^k ||\vec{x}_0 - \vec{x}||$$

A rule of thumb is that

$$||\vec{x} - \vec{x}_k|| \approx \rho(T)^k ||\vec{x}_0 - \vec{x}||$$

Question
This means in general that iterative schemes converge at what kind of rate? linear, super-linear or quadratic?

Mo' Theorems
We can denote the matrices used by each particular iterative method below:
SOR iteration uses $T_\omega = (D - \omega L)^{-1}[\omega U + (1-\omega)D]$
Jacobi Iteration uses $T_J = D^{-1}(L+U)$
Gauss-Seidel uses $T_G = (D-L)^{-1}U$

Kahan Theorem
If $a_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$ then $\rho(T_\omega) \geq |\omega - 1|$. Therefor SOR will only converge if $|\rho(T_\omega)| < 1$, or in other words, when $0 < \omega < 2$.

Ostrowski-Reich Theorem
If A is a positive definite, tridiagonal matrix then $\rho(T_G) = \rho(T_J)^2 < 1$ and the optimal choice of ω is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_J)]^2}}$$

Positive Definite Matrix
A n by n matrix A is said to be **positive definite** if A is symmetric and if $x^T Ax > 0$ for every n-dimensional column vector $x \neq 0$. A matrix is positive definite if and only if all of its eigenvalues are positive.

$\boxed{\text{GroupWork}}$

Consider the system of equations

$$\begin{array}{rcrcrcr} 4x & + & 3y & & & = & 24 \\ 3x & + & 4y & - & z & = & 30 \\ & - & y & + & 4z & = & -24 \end{array}$$

Let's try and solve this using Jacobi Iteration, Gauss-Seidel and optimal SOR. Use an initial guess of $(1,1,1)^T$. The exact solution is $(3,4,-5)^T$. Use MATLAB as a tool to assist you. You will want to use `sor.m` in the `linalg` directory of the NMM toolbox (found in `S:\Math Courses\Math370\Spring2009`).

You will need to find the spectral radius of the system, and determine whether the matrix is positive definite.