Numerical Analysis

Math 370 Spring 2009 MWF 11:30am - 12:25pm Fowler 110

 http://faculty.oxy.edu/ron/math/370/09/

Worksheet 6

SUMMARY Rates of Convergence of Iterative Sequences
CURRENT READING Mathews, p. 75

Linear, Superlinear and Quadratic Convergence of Sequences

Definition: linear convergence

Suppose we have a convergent sequence $\{x_n\}$ which converges to x_∞. If there exists a constant $0 < C < 1$ and an integer N such that

$$|x_{n+1} - x_\infty| \leq C|x_n - x_\infty|, \text{ for } n \geq N$$

we say $\{x_n\}$ converges **LINEARLY**.
In general we can say that if the following limit exists with positive constants α and λ,

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|^\alpha} = \lambda$$

then, the sequence converges at a **rate of convergence of order** α, with asymptotic error constant λ. When $\alpha = 1$ this is called **linear convergence**. When $\alpha = 2$ this is called **quadratic convergence.** If $\alpha = 1$ and $\lambda = 0$ or the following limit exists,

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_\infty|}{|x_n - x_\infty|} = 0$$

The sequence is said to converge **superlinearly**.
Let's put all of this together in the following example.

EXAMPLE

Consider $p_n = n^{-2} = \dfrac{1}{n^2}$ and $q_n = \dfrac{1}{2^n} = 2^{-n}$.

1. What is the limit of each of the sequences?

2. For each of the sequences, find out how many steps it takes to be within 10^{-4} of its limit.

3. In terms of "big oh" and "little oh" notation, can you write down a relationship between q_n and p_n?

4. Does p_n converge linearly? superlinearly? quadratically?

5. Does q_n converge linearly? superlinearly? quadratically?

6. Which sequence converges faster to its limit? Explain your answer. How is this related to their asymptotic rate of convergence?

GroupWork

Example 1 Show that $r_n = \frac{1}{n^n}$ converges superlinearly to zero.

Example 2 Show that $s_n = \frac{1}{10^{2^n}}$ converges quadratically to zero.

NOTE: Algorithms which produce sequence of approximation which converge quadratically are extremely rare.