
Numerical Analysis

Math 370 Spring 2009

MWF 11:30am - 12:25pm Fowler 110

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Class 4

SUMMARY Taylor Series Approximations and Order of Convergence for Functions

CURRENT READING Mathews & Fink, Sec 1.3

Previously we have learned the concept of “big Oh” as it applies to the rate of convergence of sequences. It can also be applied to describing the behavior of other kinds of functions as they approach other limits.

Order of Convergence of Functions

If we know that $\lim_{h \rightarrow 0} F(h) = L$ and $\lim_{h \rightarrow 0} G(h) = 0$ and if a positive constant K exists with

$$|F(h) - L| \leq KG(h), \quad \text{for sufficiently small } h$$

then we write $F(h) = L + \mathcal{O}(G(h))$

This can also be computed using the idea that $F(h) = L + \mathcal{O}(G(h))$ if and only if

$$\lim_{h \rightarrow 0} \frac{|F(h) - L|}{|G(h)|} = K$$

where K is some positive, finite constant.

EXAMPLE

Show that the expression $\cos(h) + \frac{h^2}{2}$ is $1 + \mathcal{O}(h^4)$

What is the rate of convergence of $\sin(h^3)$ as $h \rightarrow 0$

Taylor Expansions

Another approach to figuring out order of convergence of functions is to use *Taylor Series Approximations* to assist you in satisfying the inequality version of the definition.

Recall that if you have a function $f(x)$ near a point $x = a$ and $f(x)$ is infinitely-differentiable, you can write down

$$f(x) =$$

or you can truncate this series and write down

$$f(x) \approx$$

Suppose we approximate the function $f(x)$ near the point $a = 0$ then we obtain what is known as a *Maclaurin Series*.

Maclaurin Series

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$$

If we are only interested in the behavior of the function for small values of x near 0, i.e. for $|h| \ll 1$ then we can write the expression as

$$f(0 + h) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{h^k}{k!}$$

Interestingly, we can write an exact expression for the truncated form of this expression as

$$f(0 + h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2} + \mathcal{O}(h^3)$$

Yes, this last term is the same “big oh” that we have been discussing in regards to order of convergence of a function to its limit.

Exercise

Write down the following Taylor Series Approximations (for small h) for the following functions:

$$\sin(h) \approx$$

$$\cos(h) \approx$$

$$e^h \approx$$

$$(1 + h)^p \approx$$

$$\ln(1 + h) \approx$$

EXAMPLE

Show that you can use a truncated Maclaurin Expansion to prove that $\cos(h) + \frac{h^2}{2} = 1 + \mathcal{O}(h^4)$

The Three Ways Of Computing Order of Convergence of a Function are

1. **Limit Method**

2. **Bounding/Inequality Method**

3. **Truncated Taylor/Maclaurin Expansion**

(next to each of the methods above, write a short note to yourself explaining the method.)

GROUPWORK

Work in Groups of 2 or 3 to find the limit and express the order of convergence in terms of $f(h) = c + \mathcal{O}(h^\alpha) = c + o(h^\beta)$ for the following:

1. $f(h) = \frac{1 + h - e^h}{h^2}$

2. $f(h) = \frac{1}{1 - h^4}$

Truncation Error

Whenever we approximate a continuous mathematical expression by a discrete algebraic expression we incur a truncation error. For example, we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

If we do not actually evaluate the entire infinite series and stop at the 10000th term we are making a truncation error. However, thanks to “big oh” notation we have a sense of how big this truncation error is.

$$e^x = 1 + x + \mathcal{O}(x^2)$$

In general, when we use a Taylor Polynomial of degree n to approximate a function $f(x)$ near some point $x = a$ we can write

$$f(x) = P_n(x, a) + \mathcal{O}((x - a)^n), \quad \text{where } P_n(x, a) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$$

The “exact form” of the truncation error when using Taylor Polynomials can be written as

$$R_n(x) = \frac{(x - a)^{n+1}}{(n + 1)!} f^{(n+1)}(\xi)$$

where ξ is an unknown value. However you do know that $R_n(x) \rightarrow 0$ as $n \rightarrow \infty$

EXAMPLE

Example 1.16, Mathews, p 26 Given that $\int_0^1 .5e^{x^2} dx = 0.544987104184 = p$, determine the accuracy of the approximation obtained by replacing the integrand $f(x) = e^{x^2}$ with the eighth order truncated Taylor Series.