# Numerical Analysis

Math 370 Spring 2009 MWF 11:30am - 12:25pm Fowler 110

 http://faculty.oxy.edu/ron/math/370/09/

## *Class 2*

**SUMMARY** (Un)Avoidable Errors in Computing With Machines
**CURRENT READING** Mathews & Fink, Sec 1.3

**WARM-UP**

Write down the meaning of the terms **overflow**, **underflow**, **mantissa**, **characteristic**, **hexadecimal**, **bit**, **decimal machine number**, **(numerical) precision** and compare your answers with at least one other person in the class (after you have written your own!)

**RECALL**

**$k$-th digit Chopping**
In this case all the digits after $d_k$ are **ignored** ("chopped off")
**$k$-th digit Rounding**
In this case if the value of $d_{k+1} \geq 5$ then $d_k$ is replaced by $d_k + 1$
**Absolute Error**
If $\tilde{p}$ is an approximation to $p$, the **absolute error** is $|\tilde{p} - p|$
**Relative Error**
Provided $p \neq 0$, the **relative error** is $\dfrac{|\tilde{p} - p|}{|p|}$

EXAMPLE

Show that the expression involving $k$ which gives you an upper bound for the relative error involved in using chopping arithmetic is $\epsilon_{rel} = 10^{-k+1}$

It can also be shown that a bound for the relative error involved in using **rounding arithmetic** is *half* that for chopping, $\epsilon_{rel} = 0.5 \times 10^{-k+1} = 5 \times 10^{-k}$.
**Significant Digits**
The number $\tilde{p}$ is said to approximate $p$ to $k$ significant digits (or figures) if $k$ is the largest non-negative integer for which the relative error is less than $5 \times 10^{-k}$. (Mathews p. 25)

**Round-off Errors in the Quadratic Formula**

Recall that the common formula for the roots of a quadratic equation $ax^2 + bx + c = 0$ is

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and } x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Round-off error can wreak havoc with the numerical implementation of this formula. Consider

$$x^2 + 62.10x + 1 = 0$$

which has the approximate roots $x_1 = -0.01610723$ and $x_2 = -62.08390$

Because of the size of the parameters in the quadratic equation, $b^2$ is much bigger than $4ac$, so $\sqrt{b^2 - 4ac}$ is very close to $b$.

$a = 1$, $b = 62.10$, $c = 1$

$b^2 =$ $\qquad 4ac =$ $\qquad b^2 - 4ac =$ $\qquad \sqrt{b^2 - 4ac} =$

$\boxed{\text{GroupWork}}$

Using 4-digit rounding (or chopping) arithmetic compute the first root $x_1$

What's the relative error in this calculation?

Solution: change the formula for $x_1$ so that we don't have to subtract $b$ from $\sqrt{b^2 - 4ac}$

Now, a new formula for $x_1 =$

Use a similar new formula to compute $x_2$ (using 4-digit precision) and compute the relative error in $x_2$

What's the problem?

Solution: Use the new formula for $x_1$ when you have to subtract numbers which are similar in size, use the traditional formula for the other root.

When you subtract numbers of very similar size there will be a **loss of significance** or **subtractive cancellation** of digits when this operation is computed. AVOID DOING THIS!

**The Ultimate Quadratic Formula**

$$q \equiv -\frac{1}{2}\left[b + \text{sign}(b)\sqrt{b^2 - 4ac}\right]$$

where

$$\text{sign(b)} = \begin{cases} 1 & b \geq 0 \\ -1 & b < 0 \end{cases}$$

and

$$x_1 = \frac{q}{a} \qquad \text{and } x_2 = \frac{c}{q}$$