Math 370 Fall 2002 **SHOW YOUR WORK** Quiz **SIX**

1. In class we found one of the points of intersection of the hyperbola $4x^2 - y^2 = 1$ and the circle $(x-1)^2 + y^2 = 2^2$ to be (1.1165151,1.9966032).
Let $g_1(x,y) = \dfrac{8x - 4x^2 + y^2 + 1}{8}$ and $g_2(x,y) = \dfrac{2x - x^2 + 4y - y^2 + 3}{4}$ where $\vec{G}(\vec{x}) = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$

(a) *[1 pt]* Show that the fixed point(s) of the vector function $\vec{G}(\vec{x})$ are exactly the points of intersection of the hyperbola $4x^2 - y^2 = 1$ and circle $(x-1)^2 + y^2 = 4$. (HINT: one way to do this is to show algebraically that the fixed points of $\vec{G}$ satisfy the exact same equation that the points of intersection do.)

$$x = \frac{8x - 4x^2 + y^2 + 1}{8}$$
$$8x = 8x - 4x^2 + y^2 + 1$$
$$0 = -1 + 4x^2 - y^2$$
$$1 = 4x^2 - y^2 \checkmark$$

$$y = \frac{2x - x^2 + 4y - y^2 + 3}{4}$$
$$4y = 2x - x^2 + 4y - y^2 + 3$$
$$0 = -(x^2 - 2x + 1) - y^2 + 4$$
$$(x-1)^2 + y^2 = 4 \checkmark$$

(b) *[2 pts]* Starting with an initial guess of $\vec{x}_0 = (1,2)^T$ compute the next approximation to the fixed point of $\vec{G}$ using Successive Substitution, $\vec{x}_k = \vec{G}(\vec{x}_{k-1})$

$$\vec{x}_1 = \begin{pmatrix} g_1(1,2) \\ g_2(1,2) \end{pmatrix} = \begin{pmatrix} \dfrac{8\cdot 1 - 4\cdot 1^2 + 2^2 + 1}{8} \\ \dfrac{2\cdot 1 - 1^2 + 4\cdot 2 - 2^2 + 3}{4} \end{pmatrix} = \begin{pmatrix} 9/8 \\ 2 \end{pmatrix}$$

(c) *[2 pts]* Starting with an initial guess of $\vec{x}_0 = (1,2)^T$ compute the next approximation to the fixed point of $\vec{G}$ using Seidel Iteration.

$$x^{(1)} = g_1(1,2) = 9/8 \qquad y^{(1)} = g_2(9/8, 2) = \frac{2\cdot 9/8 - \left(\frac{9}{8}\right)^2 + 4\cdot 2 - 2^2 + 3}{4}$$
$$= \frac{18/8 - 81/64 + 8 - 4 + 3}{4} = 1.996$$

(d) *[2 pts]* Considering $\vec{f}(\vec{x}) = \begin{bmatrix} 4x^2 - y^2 - 1 \\ (x-1)^2 + y^2 - 2^2 \end{bmatrix}$ Find the Jacobian matrix $J(x,y)$ for the system.

$$J(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 8x & -2y \\ 2x-2 & 2y \end{pmatrix}$$

(e) *[3 pts]* Starting with an initial guess of $\vec{x}_0 = (1,2)^T$ compute the next approximation to the fixed point of $\vec{G}$ (which is also the root of $\vec{f}$) using Newton's Method.

$$J(1,2)\cdot \Delta\vec{x} = -\vec{f}(1,2)$$
$$\begin{pmatrix} 8 & -4 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = -\begin{pmatrix} 4\cdot 1^2 - 2^2 - 1 \\ 0^2 + 2^2 - 2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 1/8 & 1/8 \\ 0 & 1/4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/8 \\ 0 \end{pmatrix}$$

$$J^{-1}(1,2) = \frac{1}{8\cdot 4 - 0(-4)}\begin{pmatrix} 4 & 4 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 4/32 & 4/32 \\ 0 & 8/32 \end{pmatrix} \qquad \begin{pmatrix} x^{(1)} \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1/8 \\ 0 \end{pmatrix} = \begin{pmatrix} 9/8 \\ 2 \end{pmatrix}$$

You may have to attach/staple an extra sheet with your ~~calcu~~lations on it to support your answers.