Math 370 Fall 2004 **SHOW ALL YOUR WORK** Quiz Five

1. (a.) (*3 points*). Show that for any positive integer k, the sequence defined by $p_n = \frac{1}{n^k}$ converges **linearly** to $p_\infty = 0$.

$K > 0 \quad \lim_{n\to\infty} \frac{1}{n^K} = 0$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p_\infty|}{|p_n - p_\infty|} = \frac{\lim_{n\to\infty} \frac{1}{(n+1)^K}}{\frac{1}{n^K}}$$

$$= \lim_{n\to\infty} \frac{n^K}{(n+1)^K}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^K} = 1 > 0 \Rightarrow$$ LINEAR CONVERGENCE

(b.) (*1 point.*) For each pair of integers k and m determine a number N for which $|\frac{1}{N^k} - 0| < 10^{-m}$

$$\left|\frac{1}{N^K} - 0\right| < 10^{-m}$$

$$N^{-K} < 10^{-m}$$

$$-K \ln N < -m \ln 10$$

$$\ln N > \frac{m}{K} \ln 10 \Rightarrow N > e^{\frac{m}{K}\ln 10} = e^{\ln(10^{m/K})} = 10^{m/K}$$

2. (*4 points*) Show that $q_n = \frac{1}{10^{\alpha^n}}$ has an asymptotic rate of convergence of α! (In other words, if $\alpha = 3$, the sequence will be cubically convergent, etc.)

$$\lim_{n\to\infty} \left|\frac{q_{n+1} - 0}{|q_n - 0|^\alpha}\right| = \lim_{n\to\infty} \frac{\left|\frac{1}{10^{\alpha^{n+1}}}\right|}{\left|\frac{1}{10^{\alpha^n}}\right|^\alpha} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{(10^{-\alpha^n})^\alpha}$$

$$= \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{10^{-\alpha^n \cdot \alpha}} = \lim_{n\to\infty} \frac{10^{-\alpha^{n+1}}}{10^{-\alpha^{n+1}}} = \lim_{n\to\infty} 1 = 1$$

(b.) (*2 points*) Which element of the sequence $r_n = \frac{1}{10^{3^n}}$ will be within 10^{-12} of its limit?

$$\{r\} = \frac{1}{10^3}, \frac{1}{10^9}, \frac{1}{10^{27}}$$

r_3 is within 10^{-12} of its limit

p_N takes $10^{12/K}$ steps to get within 10^{-12} of 0