

1. Consider each of the following expressions $f(h)$ as $h \rightarrow 0$. Express each of them in the form $f(h) = L + \mathcal{O}(h^\alpha)$ with the "best" (most accurate) integer values of $\alpha > 0$. For each problem write down a value of α and L .

a. e^{h^2}

$$\lim_{h \rightarrow 0} e^{h^2} = e^0 = \boxed{1 = L}$$

$$e^\square = 1 + \square + \frac{\square^2}{2!} + \frac{\square^3}{3!} + \dots$$

$$e^{h^2} = 1 + h^2 + \frac{(h^2)^2}{2!} + \dots = 1 + \mathcal{O}(h^2) \quad \boxed{\alpha=2}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{e^{h^2} - 1}{h^2} &\stackrel{L'H}{=} \lim_{h \rightarrow 0} \frac{2he^{h^2}}{2h} \\ &\stackrel{L'H}{=} \lim_{h \rightarrow 0} e^{h^2} \\ &= 1 \checkmark \end{aligned}$$

b. $(1-h^2)^{-1}$

$$\lim_{h \rightarrow 0} \frac{1}{1-h^2} = \boxed{1 = L} \quad \boxed{\alpha=2}$$

$$\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \dots$$

$$\begin{aligned} \frac{1}{1-h^2} &= 1 + h^2 + (h^2)^2 + (h^2)^3 + \dots \\ &= 1 + h^2 + h^4 + h^6 + \dots = 1 + \mathcal{O}(h^2) \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{1-h^2} - 1 &= \lim_{h \rightarrow 0} \frac{h^2}{1-h^2} \\ &= \lim_{h \rightarrow 0} \frac{h^2}{h^2} = \lim_{h \rightarrow 0} \frac{1}{1-h^2} \\ &= 1 \checkmark \end{aligned}$$

c. $\frac{\ln(1+h)}{h}$

$$\lim_{h \rightarrow 0} \frac{\ln(1+h)}{h} = \lim_{h \rightarrow 0} \frac{1}{1+h} = \lim_{h \rightarrow 0} \frac{1}{1+h} = \boxed{1 = L} \quad \boxed{\alpha=1}$$

$$\begin{aligned} \ln(1+\square) &= \square - \frac{\square^2}{2} + \frac{\square^3}{3} - \dots \\ \ln(1+h) &= h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \dots \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\ln(1+h) - 1}{h} &= \lim_{h \rightarrow 0} \frac{\ln(1+h) - h}{h^2} \end{aligned}$$

$$\begin{aligned} \frac{\ln(1+h)}{h} &= 1 - \frac{h}{2} + \frac{h^2}{3} + \dots \\ &= 1 + \mathcal{O}(h) \end{aligned}$$

d. $\cos(h^2)$

$$\lim_{h \rightarrow 0} \cos(h^2) = \boxed{1 = L} \quad \boxed{\alpha=4}$$

$$\cos(\square) = 1 - \frac{\square^2}{2!} + \frac{\square^4}{4!} - \dots$$

$$\begin{aligned} \cos(h^2) &= 1 - \frac{(h^2)^2}{2!} + \frac{(h^2)^4}{4!} - \dots \\ &= 1 - \frac{h^4}{2} + \frac{h^8}{24} - \dots = 1 + \mathcal{O}(h^4) \end{aligned}$$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\cos(h^2) - 1}{h^4} &= \lim_{h \rightarrow 0} \frac{-2h \sin(h^2)}{4h^3} \\ &= \lim_{h \rightarrow 0} \frac{\sin(h^2)}{-2(h^2)} \end{aligned}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{1}{1+h} - 1 \\ &= \lim_{h \rightarrow 0} \frac{-1}{(1+h)^2} = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{2h \cos(h^2)}{-4h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(h^2)}{-2} = -\frac{1}{2} \end{aligned}$$

e. $1 + \sin(h^3)$

$$\lim_{h \rightarrow 0} 1 + \sin(h^3) = \boxed{1 = L}$$

$$\begin{aligned} 1 + \sin(h^3) &= 1 + h^3 - \frac{h^9}{6} + \dots \\ &= 1 + \mathcal{O}(h^3) \end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{1 + \sin(h^3) - 1}{h^3}$$

$$\sin \square = \square - \frac{\square^3}{3!} + \frac{\square^5}{5!} - \dots \quad \boxed{\alpha=3}$$

$$\sin(h^3) = (h^3) - \frac{(h^3)^3}{3!} + \frac{(h^3)^5}{5!} - \dots = h^3 - \frac{h^9}{6} + \dots$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\sin(h^3)}{h^3} \\ &= \lim_{h \rightarrow 0} \frac{3h^2 \cos(h^3)}{3h^2} \\ &= \lim_{h \rightarrow 0} \cos(h^3) = 1 \end{aligned}$$