Test 1: Numerical Analysis

Math 370 Spring 2009 | Friday February 27

 | 11:30am-12:25pm

Name: ________________________________

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, limited-notes, closed book, test. **You may use one 8.5" by 11" sheet of paper with handwritten information on one side.** It should be stapled to your exam when you turn it in. **Use of calculators is allowed.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

Pledge: I, ____________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		30
2		40
3		30
BONUS		10
Total		**100**

1. *[30 points total.]* **Order Notation and Taylor Series.**

Consider the integrally-defined function $F(h) = 1 - h + \int_0^h e^{-x^2} dx$

GOAL: to describe the behavior of this given function $F(h)$ for very small values of h, i.e. $|h| << 1$, or as $h \to 0$, using the Bachmann-Landau symbol $\mathcal{O}$.

(a) *[6 points].* Write down the first three non-zero terms of the Maclaurin Series for $f(x) = e^{-x^2}$.

(b) *[6 points].* Use your answer in part (a) to write down the first three non-zero terms of the Maclaurin series for $F(h)$.

(c) *[6 points].* Use your previous answers to show that you can write down the behavior of the integrally-defined function as $F(h) = L + \mathcal{O}(h^p)$ (Give values for L and p).

(d) *[12 points].* Use L'Hôpital's Rule to confirm your choices of L and p in part (c) by using the limit definition of "big oh."

[NOTE: The Fundamental Theorem of Calculus, The Limit Defintion of "Big Oh" and the First Three Terms of the McLaurin Exapansion of e^x can be bought for 3 points each!]

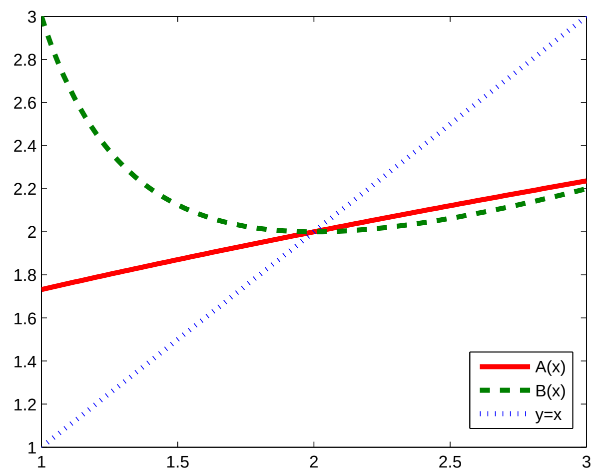
2. *[40 points total.]* **Sequences, Limits, Picard Iteration, Convergence Criteria.**

GOAL: to determine the limit of two functional iterations and the relative speed of convergence of two different iteration schemes from given Picard Iteration output, analysis, calculus and/or graphical info.

Consider the two sequences $x_{n+1} = \sqrt{2 + x_n} = A(x_n)$ and $x_{n+1} = \dfrac{x_n^2 + 2}{2x_n - 1} = B(x_n)$ with $x_0 = 1$. Note $\sqrt{x} \geq 0 \quad \forall x \geq 0$.

(a) *[10 points].* What are the values of $\lim_{n\to\infty} A(x_n)$ and $\lim_{n\to\infty} B(x_n)$? EXPLAIN YOUR ANSWER.

(b) *[10 points].* Consider the graph of $A(x) = \sqrt{2 + x}$ and $B(x) = \dfrac{x^2 + 2}{2x - 1}$ given below, together with the line $y = x$ on the interval $1 \leq x \leq 3$. Which of the following sequences will converge faster to their limit, $x_{n+1} = A(x_n)$ or $x_{n+1} = B(x_n)$? EXPLAIN YOUR ANSWER.



(c) *[10 points].* One of the sequences $A(x_n)$ or $B(x_n)$ is quadratically convergent to its limit while the other exhibits linear convergence. Prove which sequence is linearly convergent, and which sequence is *faster* than linearly convergent. EXPLAIN YOUR ANSWER.

(d *[10 points].* Consider the data given in the table below for functional iteration run repeatedly on Method 1 and Method 2. Identify which method corresponds to which iterative scheme, i.e. $x_{n+1} = A(x_n)$ or $x_{n+1} = B(x_n)$. Also discuss the stopping criterion that must have been used to generate the following data. Where would each method have stopped if the stopping criterion had been accuracy to within 5 decimal places? EXPLAIN YOUR ANSWER(S).

Method 1

n	$g(x_n)$	x_n	$\|x_n - x_{n-1}\|$
⋮	⋮	⋮	⋮
2	2.012e+000	2.200000000000000	1.88235e-001
3	2.000e+000	2.011764705882353	1.17189e-002
4	2.000e+000	2.000045777065690	4.57763e-005
5	2.000e+000	2.000000000698492	6.98491e-010
6	2.000e+000	2.000000000000000	0.00000e+000
7	2.000e+000	2.000000000000000	0.00000e+000
8	2.000e+000	2.000000000000000	0.00000e+000
9	2.000e+000	2.000000000000000	0.00000e+000

Method 2

n	$g(x_n)$	x_n	$\|x_n - x_{n-1}\|$
⋮	⋮	⋮	⋮
2	1.983e+000	1.931851652578137	5.10380e-002
3	1.996e+000	1.982889722747621	1.28281e-002
4	1.999e+000	1.995717846477207	3.21132e-003
5	2.000e+000	1.998929174952731	8.03100e-004
6	2.000e+000	1.999732275819124	2.00792e-004
7	2.000e+000	1.999933067834802	5.01990e-005
8	2.000e+000	1.999983266888701	1.25498e-005
9	2.000e+000	1.999995816717800	3.13746e-006

3. *[30 points total.]* **TRUE or FALSE.**

INSTRUCTIONS: Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is FALSE providing a counter example for which the statement is not true is best. If you think the answer is TRUE you should also explain why you believe the statement.

(a) TRUE OR FALSE? "If $\{p_n\} = 2^{-n}$ is linearly convergent to $p = 0$ and $\{q_n\} = 10^{-(2^n)}$ is quadratically convergent to $q = 0$ then $\lim_{n\to\infty} \frac{q_n - q}{p_n - p} = K, \qquad K > 0$."

(b) TRUE OR FALSE? "The MATLAB commands `x=linspace(0,1,10)` and `x=[0:.1:1]` will produce exactly the same output."

(c) TRUE OR FALSE? "The **machine precision** i.e., the number ϵ_m such that $1+\epsilon_m = 1$, is the same on all computers."

BONUS QUESTION. Order of Convergence, Error and Recursive Difference Equations. (*10 points.*)
Consider the recursive error sequence $|e_{n+1}| = \lambda |e_n|^p$ where $\lambda > 0$,$e_0 = A$ and $\lim_{n\to\infty} e_n = 0$. You can assume all values of e_n are positive. Show that an explicit expression for e_n in terms of λ, A, p and most importantly n can be obtained which is

$$e_n = \lambda^{\left(\sum_{k=0}^{n} p^k\right)} A^{(p^n)}.$$

Confirm that this given error sequence converges to zero with asymptotic order of convergence p and asymptotic rate constant λ.