Test 2: Numerical Analysis

Math 370 Friday, November 19, 2004

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Directions: Read *ALL* 3 (three) problems before answering any of them because they are related. This is a one hour, closed notes, closed book, test. This test has 6 pages. There is this coversheet, 4 pages of questions and the last page is a graph. You must show all relevant work to support your answers. Use complete English sentences and clearly indicate your final answer from your "scratch work."

No.	Score	Maximum
1		30
2		50
3		20
TOTAL		100

1. *[30 pts. total]* **Iterative Solution of Nonlinear Systems: Theory.**

The goal of this problem is for you to illustrate your understanding of Newton's Method applied to nonlinear systems by modifying it in two different ways. Recall that Newton's Method applied to $\begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \vec{f}(\vec{x}) = \vec{0}$ can be written in Fixed Point Iteration form as

$\vec{x}^{(k+1)} = \vec{G}(\vec{x}^{(k)})$ or $\begin{bmatrix} x^{(k+1)} \\ y^{(k+1)} \end{bmatrix} = \begin{bmatrix} g_1(x^{(k)}, y^{(k)}) \\ g_2(x^{(k)}, y^{(k)}) \end{bmatrix}$ and $\vec{x}^{(0)} = \begin{bmatrix} x^{(0)} \\ y^{(0)} \end{bmatrix}$ where

$$g_1(x,y) = x - \frac{f_1(x,y)\frac{\partial f_2}{\partial y}(x,y) - f_2(x,y)\frac{\partial f_1}{\partial y}(x,y)}{\det(J(x,y))}$$

$$g_2(x,y) = y - \frac{f_2(x,y)\frac{\partial f_1}{\partial x}(x,y) - f_1(x,y)\frac{\partial f_2}{\partial x}(x,y)}{\det(J(x,y))}$$

and $\det J(x,y) = \frac{\partial f_1}{\partial x}(x,y)\frac{\partial f_2}{\partial y}(x,y) - \frac{\partial f_1}{\partial y}(x,y)\frac{\partial f_2}{\partial x}(x,y)$

(a) *[20 points]*. Using similar notation as above, write down a functional iteration formula which would represent computing the $\vec{x}^{(k+1)}$ approximation from the $\vec{x}^k$ approximation, using **Lazy Newton's Method** (also known as the "Fixed Jacobian" Method) for the same problem $\vec{f}(\vec{x}) = \vec{0}$. **Be very specific to indicate which iterate is inputted into which function.**

$$x^{(k+1)} = x^{(k)} - \frac{f_1(x^{(k)},y^{(k)})\frac{\partial f_2}{\partial y}(x^{(0)},y^{(0)}) - f_2(x^{(k)},y^{(k)})\frac{\partial f_1}{\partial y}(x^{(0)},y^{(0)})}{\det J(x^{(0)},y^{(0)})}$$

$$y^{(k+1)} = y^{(k)} - \frac{f_2(x^{(k)},y^{(k)})\frac{\partial f_1}{\partial x}(x^{(0)},y^{(0)}) - f_1(x^{(k)},y^{(k)})\frac{\partial f_2}{\partial x}(x^{(0)},y^{(0)})}{\det J(x^{(0)},y^{(0)})}$$

The J and all partials are evaluated at $\vec{x}^{(0)}$.

(b) *[10 points]*. Use your Lazy Newton's formula in (a) to write down a specific formula for how to compute the third iterate $\vec{x}^{(3)}$ from the second iterate $\vec{x}^{(2)}$.

$$x^{(3)} = x^{(2)} - \frac{f_1(\vec{x}^{(2)})\frac{\partial f_2}{\partial y}(\vec{x}^{(0)}) - f_2(\vec{x}^{(2)})\frac{\partial f_1}{\partial y}(\vec{x}^{(0)})}{\det J(x^{(0)},y^{(0)})}$$

$$y^{(3)} = y^{(2)} - \frac{f_2(\vec{x}^{(2)})\frac{\partial f_1}{\partial x}(\vec{x}^{(0)}) - f_1(\vec{x}^{(2)})\frac{\partial f_2}{\partial x}(\vec{x}^{(0)})}{\det J(\vec{x}^{(0)})}$$

2. *[50 pts. total]* **Iterative Solution of Nonlinear Systems: Application.** Consider the nonlinear system

$$f_1(x,y) = y^2 - x = 0$$
$$f_2(x,y) = xy - 1 = 0$$

(a) *[30 points]*. Let $\vec{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and compute $\vec{f}(\vec{x}^{(0)})$ and $J(\vec{x}^{(0)})$ as well as $||\vec{f}(\vec{x}^{(0)})||$ and $||J(\vec{x}^{(0)})||$ (use the same norm for the last two).

$$\vec{f}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} f_1(1,0) \\ f_2(1,0) \end{pmatrix} = \begin{pmatrix} 0^2 - 1 \\ 0\cdot 1 - 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$J(x,y) = \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = \begin{pmatrix} -1 & 2y \\ y & x \end{pmatrix}$$

$$J\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$||\vec{f}(\vec{x}^{(0)})||_1 = |-1| + |-1| = 2$$

$$||f(\vec{x}^{(0)})||_2 = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

$$||f(\vec{x}^{(0)})||_\infty = \max(|-1|, |-1|) = 1$$

$$||J(\vec{x}^{(0)})||_1 = \text{max col sums} = \max(1, 1) = 1$$

$$||J(\vec{x}^{(0)})||_\infty = \text{max row sums} = \max(1, 1) = 1$$

$$||J(\vec{x}^{(0)})||_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} = 1$$

(b) *[20 points]*. Starting with $\vec{x}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ compute the second iterate $\vec{x}^{(2)}$ using Lazy Newton's Method.

$$\vec{x}^{(1)} = \vec{x}^{(0)} - J^{-1}(\vec{x}^{(0)})\, \vec{f}(\vec{x}^{(0)})$$

$$J^{-1}(\vec{x}^{(0)}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = J(\vec{x}^{(0)})$$

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \leftarrow \text{same as Newton's}$$

$$\vec{f}(\vec{x}^{(1)}) = \begin{pmatrix} f_1(0,1) \\ f_2(0,1) \end{pmatrix} = \begin{pmatrix} 1^2 - 0 \\ 1 \cdot 0 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{x}^{(2)} = \vec{x}^{(1)} - J^{-1}(\vec{x}^{(0)})\, \vec{f}(\vec{x}^{(1)})$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{x}^{(2)}}$$

3. *[20 pts. total]* **Iterative Solution of Nonlinear Systems: Analysis.**
Here are the numerical results solving the system using Newton's Method for Systems.

Newton's Method Results

k	$x^{(k)}$	$y^{(k)}$
0	1.00000000	0.00000000
1	0.00000000	1.00000000
2	1.00000000	1.00000000
3	1.00000000	1.00000000
$\vdots$	$\vdots$	$\vdots$

(a) *[5 points].* Confirm that the nonlinear system has one solution, at $(1, 1)$.

$$\vec{f}(1,1) = \begin{pmatrix} f_1(1,1) \\ f_2(1,1) \end{pmatrix} = \begin{pmatrix} 1^2 - 1 \\ 1\cdot 1 - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$(1,1)$ is a root of $\vec{f}(\vec{x})$.

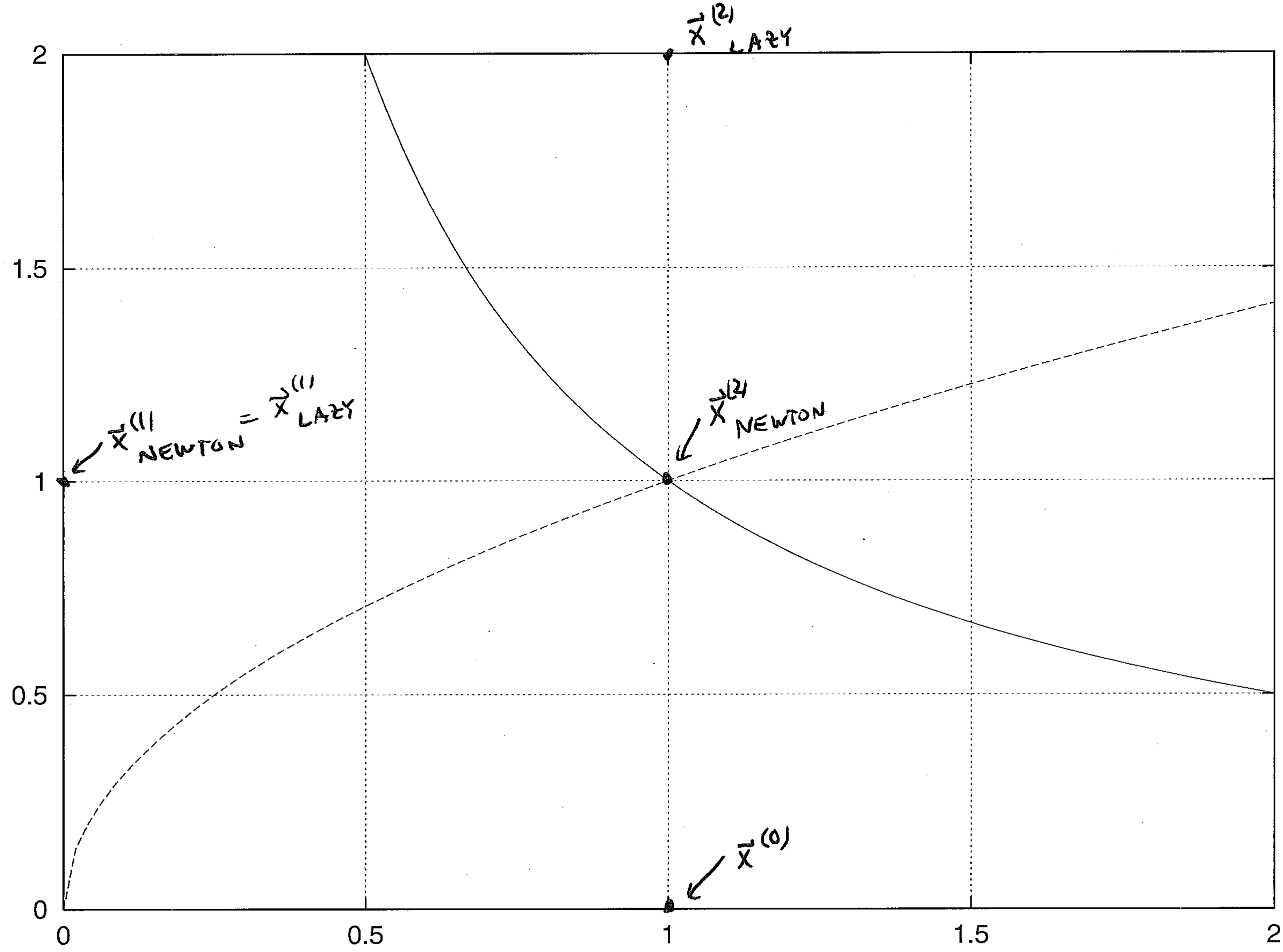
(b) *[10 points].* Choose your favorite norm and compute the error between the exact solution and each of your approximations using Lazy Newton's Method (your answers from **2(b)**).

$$\|\vec{e}^{(1)}\| = \left\|\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| = \left\|\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\| = 1$$

$$\|\vec{e}^{(2)}\| = \left\|\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\| = 1$$

(c) *[5 points].* Which method (Newton or Lazy Newton) produces the most accurate estimate to the exact solution of the given system after 2 iterations? Give a reason which explains your results.

Newton's gets the exact answer in two steps.



2
1.5
1
0.5
0
0
0.5
1
1.5
2
$\vec{x}^{(2)}_{LAZY}$
$\vec{x}^{(1)}_{NEWTON} = \vec{x}^{(1)}_{LAZY}$
$\vec{x}^{(2)}_{NEWTON}$
$\vec{x}^{(0)}$