Numerical Analysis

Math 370 Fall 2002 MWF 9:30am - 10:25pm
 Fowler 127

Worksheet 25

SUMMARY Introduction to Piecewise Polynomial Interpolation
READING Recktenwald, pp. 521–538

Error due to Interpolation
Recall the error formula for Lagrange Interpolation looks like this:

$$e(x) = f(x) - P(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $\xi(x)$ is in (a, b)

Of course, usually we don't know what function we are interpolating (we may just have raw data), but when we do, we can use the above error formula. (Of course if we knew the function it's not very likely that we would actually interpolate to get a function value.) It is still useful to analyze this error formula because it provides us with insight into deeper features of interpolation.

The term $f^{(n+1)}(\xi(x))$ represents an unknown constant. What is usually done is that one tries to bound it by a known number M, so that one gets an error bound for the maximum error one is making in doing the interpolation.
Example
2. Suppose we are trying to use a linear interpolating polynomial $P_1(x)$ for $f(x)$ at x_0 and x_1. Assume that $f''(x) \leq M$ for $x \in [x_0, x_1]$. We can use the error formula to write down an expression for the error $|(f(x) - p(x)|$ below:

3. What happens to the error if x is not in the interval $[x_0, x_1]$?
What does this tell you about the error of computing an **extrapolated** value of $f(x)$ compared to the error of computing an **interpolated** value?

4. Let's try and get an error bound from interpolating by computing the maximum of the function $|(x - x_0)(x - x_1)|$

Thus, $e(x) = |f(x) - P_1(x)| \leq$

Application

5. Suppose that we want to compute "cheap and dirty" values of the sine function by just evaluating at a number of equally spaced points and then using linear interpolation to compute intermediate values. How small would the spacing h between the points have to be to achieve an accuracy of four digits?

Piecewise Polynomial Interpolation

Instead of using one interpolating polynomial of high degree, there is also the idea of using a lot of polynomials of small degree to do interpolation. For example, with the example of producing cheap and dirty sines, we said that we would use a number of linear interpolations between consecutive nodes. We say that these are **piecewise polynomials** because the "pieces" (the individual linear interpolating polynomials) are tied together at the "knots" (also known as the nodes). These pieces are also known as **splines**.

Piecewise Linear Interpolation

7. Sketch a figure of piecewise linear interpolation for the following data

8. Given accurate data (x_k, y_k) for $k = 0$ to n come up with the equation for a typical piecewise linear interpolant $s_k(x)$

9. What is the relationship between $s_k(x)$ on the kth interval and s_{k+1} on the $(k+1)$th interval? [HINT: think about their common point.]

10. What are some drawbacks of doing piecewise linear interpolation instead of doing piecewise polynomial interpolation with a higher degree polynomial?

The most common piecewise polynomial interplation is done with polynomials of degree **three**, thus leading to the famous **cubic spline** problem.