Numerical Analysis

Math 370 Fall 2002 MWF 9:30am - 10:25pm
 Fowler 127

Worksheet 18

SUMMARY Analysis of Iterative Methods for Solving Linear Systems

Consider the system

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15 \end{aligned}$$

We can re-write these equations as

$$x^{(k+1)} = \frac{7 + y^{(k)} - z^{(k)}}{4}, \qquad y^{(k+1)} = \frac{21 + 4x^{(k)} + z^{(k)}}{8}, \qquad z^{(k+1)} = \frac{15 + 2x^{(k)} - y^{(k)}}{5}$$

OR

$$x^{(k+1)} = \frac{7 + y^{(k)} - z^{(k)}}{4}, \qquad y^{(k+1)} = \frac{21 + 4x^{(k+1)} + z^{(k)}}{8}, \qquad z^{(k+1)} = \frac{15 + 2x^{(k+1)} - y^{(k+1)}}{5}$$

Which of these schemes represents Gauss-Seidel Iteration and which represents Jacobi Iteration?

Can you generalize these schemes if the linear system looks like:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

Write down the general iterative formula for Jacobi Iteration on a 3x3 system here:

Write down the general iterative formula for Gauss-Seidel Iteration on a 3x3 system here

Let's try to really generalize these schemes when applied to a system of n equations in n variables.

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\
\vdots &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}$$

Write down the general form of the iterative scheme for Jacobi Iteration using a_{ij}

We can use information from Jacobi Iteration to derive the scheme for Gauss-Seidel Iteration

Matrix representation of iterative schemes for linear systems
We have written down the iterative scheme implementation of Jacobi and Gauss-Seidel iteration but the more useful way to think about these schemes is using the matrix representation of the generic iterative scheme

$$\underline{x}^{(k+1)} = T\underline{x}^{(k)} + \underline{c}$$

and we'll derive how T depends on A and $\vec{c}$ depends on A and $\vec{b}$ for each method.
We will write the matrix A as the sum of three matrices D (diagonal matrix), L (lower triangular) and U (upper triangular) such that

$$A = D - L - U$$

For example, write down D, L and U for the original linear system on page 1

The system $A\underline{x} = \underline{b}$ can be written as

$$\begin{aligned}
(D - L - U)\underline{x} &= \underline{b} \\
D\underline{x} &= L\underline{x} + U\underline{x} + \underline{b} \\
\underline{x} &= D^{-1}(L + U)\underline{x} + D^{-1}\underline{b} \\
\underline{x}^{(k+1)} &= D^{-1}(L + U)\underline{x}^{(k)} + D^{-1}\underline{b}
\end{aligned}$$

Another choice is

$$\begin{aligned}
(D - L - U)\underline{x} &= \underline{b} \\
(D - L)\underline{x} &= U\underline{x} + \underline{b} \\
\underline{x} &= (D - L)^{-1}U\underline{x} + (D - L)^{-1}\underline{b} \\
\underline{x}^{(k+1)} &= (D - L)^{-1}U\underline{x}^{(k)} + (D - L)^{-1}\underline{b}
\end{aligned}$$

Which of the above schemes represents Jacobi Iteration and which represents Gauss-Seidel? How can you tell?

MATLAB implementation

We have looked at methods for finding iterative solutions of systems of linear equations. The methods we know are Jacobi Iteration (`jacobi.m`) and Gauss-Seidel (`gseidel.m`) which can be found in the `Math Courses/Math370/Fall2002/nmm/linalg`.

Use Gauss-Seidel and Jacobi Iteration to solve the linear system

$$\begin{aligned} 2x + 8y - z &= 11 \\ 5x - y + z &= 10 \\ -x + y + 4z &= 3 \end{aligned}$$

with different initial guesses. Do you EXPECT the system as currently constituted to converge using Jacobi and/or Gauss-Seidel Iteration? (HINT: recall the conditions on diagonal dominance for convergence of iterative methods to the solution to $A\underline{x} = \underline{b}$.)

What could you do to change the system so that you could use the iterative methods to generate a solution to the system?