Version: November 20, 2002
Due Date: **Wednesday December 11, 5:00pm**

1 Background

The flow of a fluid (liquid or gas) is governed, in general, by a system of coupled, non-linear partial differential equations. We shall consider the simplified case of modelling an inviscid (no viscosity), irrotational (no wakes, boundary layers or vortices) and incompressible (density doesn't change with change in pressure) fluid. For this specialized case, which occurs in a wide number of actual physical situations (for example flow of air of around comercial aircraft, flow of water in cylindrical pipes, et cetera), the dynamics of the fluid are described by *potential theory.* Specifically, in this project we will be looking at two-dimensional, incompressible potential flow.

Our goal is to write down boundary value problems (which consist of a partial differential equation combined with boundary conditions) for a number of classical flow situations, and to solve these problems numerically. We shall consider a problem solved when the difference between our computed solution and the exact solution is less than some tolerance. We can then produce a contour map of the converged numerical solution describing the fluid flow throughout the region of interest.

2 The Model: Potential Theory

Potential Theory assumes that a velocity potential ϕ exists such that the velocity of the fluid at any point can be obtained by computing $\vec{v} = \nabla\phi$. In cartesian coordinates, this means that the velocity at any point in the plane (x, y) is given by

$$\vec{v} = u\hat{x} + v\hat{y} = \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y}. \tag{1}$$

In polar coordinates, the velocity can be found at any point (r, θ) by

$$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}, \tag{2}$$

where v_r and v_θ are the components in the radial and angular directions.

In addition to the velocity potential function, there is another function which assists in the description of fluid flow: the stream function, ψ. The stream function is defined using the equations

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}. \tag{3}$$

In potential flow, the points in the plane at which the velocity potential is constant are curves called *equipotentials.* When the stream function is constant the corresponding curves are called *streamlines* and represent the path that particles will travel in the flow. Particles will travel parallel to streamlines and thus this means the flow will not cross them. Therefore streamlines can represent flow boundaries. This idea is important in mathematically describing the flow around differently-shaped objects.

The governing equation for two dimensional, incompressible flow is Laplace's Equation

$$\nabla^2\phi = 0. \tag{4}$$

In cartesian coordinates this becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \phi_{xx} + \phi_{yy} = 0. \tag{5}$$

In polar coordinates Laplace's Equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{r}\left(r\phi_r\right)_r + \frac{1}{r^2}\phi_{\theta\theta} = 0. \tag{6}$$

From the multiple definition of the velocity components in (3) and (1) one can see that the stream function and the velocity potential can be mathematically related to each other. The equations which relate the two functions are known as the Cauchy-Riemann equations,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{7}$$

And, the corresponding version of the Cauchy-Riemann equations in polar coordinates are

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}. \tag{8}$$

From Complex Analysis it is known that functions which satisfy the Cauchy-Riemann Equations are *harmonic functions*, which means that they also satisfy Laplace's Equation. In other words not only does the velocity potential satisfy (5), but so does the stream function:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{9}$$

Boundary Value Problems

The problem of describing the flow of a fluid in a region mathematically is one of solving a boundary value problem for either ϕ and/or ψ (if you have one, you can get the other). Let us consider a rectangular domain in the xy-plane. A solution of the problem would be a functional representation of $\psi(x, y)$ which we could differentiate and thus give the velocity components $u(x, y)$ and $v(x, y)$ at every point in our domain. We know that $\phi(x, y)$ and $\psi(x, y)$ satisfy Laplace's Equation at every point in the interior of the domain, but we also need to have equations for ϕ or ψ on the boundary of the domain. These equations are called *boundary conditions*. This is very similar to the idea that when solving *initial value problems* one needs not only a differential equation but an initial condition to obtain a unique solution. With *boundary value problems* one needs a partial differential equation and boundary conditions to produce a unique solution.

3 The Problem: Theoretical Fluid Mechanics

The term project will be to mathematically describe and compute solutions for classic theoretical fluid mechanic scenarios. These will be Fluid Around a Corner, Fluid Into A Semi-Infinite Channel, Fluid Around An α-Wedge, and Fluid Past A Circular Object (Extra Credit).

Situation A: Flow Around A Corner

Consider Figure 1. The problem of incompressible inviscid flow around a 90° corner (at the origin) can be written as a boundary value problem for the stream function $\psi(x, y)$. Let the speed of the fluid down into the corner be unity, and equal to the speed of the flow to the right exiting the corner. Let's derive the boundary condition for this situation.

The governing partial differential equation for all our situations is Laplace's Equation

$$\nabla^2 \psi = 0$$

Along $y = 0$ there is a wall, and the flow will be directly vertical downwards, in other words $v(x, 0) = -\psi_x(x, 0) = -1$ and $u(x, 0) = \psi_y = 0$. So, since $u = \psi_y$ and $v = -\psi_x$ we know that

$$-\frac{\partial \psi}{\partial x} = -1 \Rightarrow \psi(x, 0) = x$$

Similarly, as the flow leaves the corner it is completely horizontal, so $v(1, y) = -\psi_x(1, y) = 0$ and $u(1, y) = \psi_y(1, y) = 1$, which means that

$$\frac{\partial \psi}{\partial y} = 1 \Rightarrow \psi(1, y) = y$$

The boundaries of the flow (i.e. the wall and the floor) are represented by the constant value of the stream function along the y-axis and x-axis. The value of the constant is usually taken to be zero along flow boundaries.

$$\psi(x, 0) = \psi(0, y) = 0$$

The exact solution in this case is $\psi(x, y) = xy$.

Situation B: Flow at the End of a Semi-Infinite Channel

Consider Figure 2. Write down a boundary value problem for the stream function for the fluid flow into a channel which has walls at $x = -\pi/2$ and $x = \pi/2$ and a floor at $y = 0$.
The exact solution for this problem is $\psi(x, y) = \sinh(y)\cos(x)$.

Situation C: Flow Into An α-Wedge

Consider changing Situation A so that instead of the corner being a 90° it is α radians. Write down a boundary value problem for the stream function of the flow into the "α-Wedge." The main feature of this problem is that it is more easily expressed in polar coordinates. Also note that Situation A is a special case of Situation C (with $\alpha = \pi/2$). Can you show this? (NOTE: that if $f(x, y)$ is a solution of Laplace's Equation, so is $Kf(x, y)$, where K is a constant.

The exact solution is $\psi(r, \theta) = r^{\pi/\alpha} \sin\left(\frac{\pi\theta}{\alpha}\right)$.

Situation D: Flow Past A Cylinder

Write down the boundary value problem for flow past a cylinder of unit radius at the origin.

The exact solution of this problem is $\psi(r,\theta) = \left(r - \frac{1}{r}\right)\sin(\theta)$

4 Computational Fluid Dynamics

Approximating Laplace's Equation Numerically

Let us say we are only interested in a portion of the xy-plane which we call $\mathcal{D} : (x, y) \in a \leq x \leq b, c \leq y \leq d$. Let us partition the horizontal (x-axis) and vertical (y-axis) coordinates, into m and n pieces, respectively. Thus we have changed our region of interest from the infinite number of points in the plane to the finite number of $m \times n$ discrete points. This process is called *discretization.*

We can write formulas for the precise points in the plane which we are considering. They all have the form (x_i, y_j) where i ranges from 1 to m and j ranges from 1 to n.

$x_i = a+(i-1)\Delta x$ where $\Delta x = \dfrac{b-a}{m-1}$ is the separation between neighbouring points (on the x-axis). Similarly, $y_j = c + (j-1)\Delta y$ where $\Delta y = \dfrac{d-c}{n-1}$.

Our numerical problem will be to find the value of the stream function at each of these discrete points. We will thus need to produce $m \times n$ equations for $m \times n$ variables. We shall denote these variables as $\psi_{i,j}$, where $\psi_{i,j} = \psi(x_i, y_j)$.

The derivatives in the Laplacian can be approximated using finite differences. This involves using the idea that

$$f''(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x} + \mathcal{O}((\Delta x)^2).$$

Thus in the case of the second derivatives of the stream function in Laplace's Equation we can use this idea to produce an approximation for ψ_{xx} at each (x_i, y_j),

$$\frac{\partial^2\psi}{\partial x^2} \approx \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2}. \tag{10}$$

You should be able to show that at the internal parts of the domain the discrete form of Laplace's Equation is:

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{(\Delta y)^2} = 0$$

This can be re-arranged to form an explicit equation for $\psi_{i,j}$,

$$\left(\frac{\Delta y}{\Delta x}\right)^2 (\psi_{i+1,j} + \psi_{i-1,j}) + \psi_{i,j+1} + \psi_{i,j-1} - 2\left(\left(\frac{\Delta y}{\Delta x}\right)^2 + 1\right)\psi_{i,j} = 0 \tag{11}$$

$$\psi_{i,j} = \frac{\left(\frac{\Delta y}{\Delta x}\right)^2 (\psi_{i+1,j} + \psi_{i-1,j}) + \psi_{i,j+1} + \psi_{i,j-1}}{2\left(\frac{\Delta y}{\Delta x}\right)^2 + 1}$$

If $\Delta x = \Delta y$ then the equation becomes

$$\psi_{i,j} = \frac{\psi_{i,j+1} + \psi_{i,j-1} + \psi_{i+1,j} + \psi_{i-1,j}}{4}$$

Notice that Laplace's Equation will only be the governing equation for $(n-2)(m-2) = nm - 2n - 2m + 4$ of the variables. The boundary conditions will take care of $2n + 2(m-2) = 2n + 2m - 4$ equations. Together, you will have $n \times m$ equations in $n \times m$ unknowns.

For example, for $n = m = 5$ the set of $(n-2) \times (m-2) = 9$ equations for the 9 internal $\psi_{i,j}$ values will be

$$\begin{aligned}
-4\psi_{2,2} + \psi_{1,2} + \psi_{3,2} + \psi_{2,3} + \psi_{2,1} &= 0 \\
-4\psi_{3,2} + \psi_{2,2} + \psi_{4,2} + \psi_{3,3} + \psi_{3,1} &= 0 \\
-4\psi_{4,2} + \psi_{3,2} + \psi_{5,2} + \psi_{4,3} + \psi_{4,1} &= 0 \\
-4\psi_{2,3} + \psi_{2,4} + \psi_{2,2} + \psi_{1,3} + \psi_{3,3} &= 0 \\
-4\psi_{3,3} + \psi_{3,4} + \psi_{3,2} + \psi_{2,3} + \psi_{4,3} &= 0 \\
-4\psi_{4,3} + \psi_{4,4} + \psi_{4,2} + \psi_{3,3} + \psi_{5,3} &= 0 \\
-4\psi_{2,4} + \psi_{2,5} + \psi_{2,3} + \psi_{3,4} + \psi_{1,4} &= 0 \\
-4\psi_{3,4} + \psi_{3,5} + \psi_{3,3} + \psi_{4,4} + \psi_{2,4} &= 0 \\
-4\psi_{4,4} + \psi_{4,5} + \psi_{4,3} + \psi_{5,4} + \psi_{3,4} &= 0
\end{aligned}$$

There are $2n + 2m - 4 = 16$ boundary conditions (for $n = m = 5$). Along the y-axis, there is a wall

$$\psi_{1,2} = \psi_{1,3} = \psi_{1,4} = \psi_{1,5} = 0$$

Along the x-axis, there is another boundary

$$\psi_{2,1} = \psi_{3,1} = \psi_{4,1} = \psi_{5,1} = 0$$

And at the intersection (corner)

$$\psi_{1,1} = 0$$

At the input and output parts of

$$\begin{aligned}
\psi_{5,2} &= x_5 y_2 \\
\psi_{5,3} &= x_5 y_3 \\
\psi_{5,4} &= x_5 y_4 \\
\psi_{5,5} &= x_5 y_5 \\
\psi_{2,5} &= x_2 y_5 \\
\psi_{3,5} &= x_3 y_5 \\
\psi_{4,5} &= x_4 y_5
\end{aligned}$$

To avoid confusion, it might be useful to rename the unknown variables $\psi_{i,j}$ as ψ_{ij}. In matrix form, the equations can be written as

$$\begin{pmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{pmatrix}
\begin{pmatrix}
\psi_{22} \\ \psi_{32} \\ \psi_{42} \\ \psi_{23} \\ \psi_{33} \\ \psi_{43} \\ \psi_{24} \\ \psi_{34} \\ \psi_{44}
\end{pmatrix}
=
\begin{pmatrix}
-\psi_{21} - \psi_{12} \\ -\psi_{31} \\ -\psi_{52} - \psi_{41} \\ -\psi_{13} \\ 0 \\ -\psi_{53} \\ -\psi_{14} - \psi_{25} \\ -\psi_{35} \\ -\psi_{45} - \psi_{54}
\end{pmatrix}$$

Notice that you can think of the matrix in this equation as having the form

$$\begin{pmatrix}
A & I & 0 \\
I & A & I \\
0 & I & A
\end{pmatrix} \vec{x} = \vec{b}$$

where A and I are 3×3 matrices (also known as "blocks" or block matrices). In other words, the matrix in question is block tri-diagonal, symmetric and positive definite. Thus if we use SOR we can find an optimal ω parameter which accelerates convergence.

We shall consider the problem solved when the difference between the exact solution and the computed solution is small. One choice is to have $\max_{1\le i\le m, 1\le j\le n} |\psi_{i,j} - \psi(x_i, y_j)| < \epsilon$ where ϵ is about 10^{-3}.

4.1 Using Gauss-Seidel and SOR To Solve The System

There is more than one way to implement the solution of the mn linear equations. One could use the given `sor.m` and `gseidel.m` and even `jacobi.m` routines. But then you would have to come up with a way to compute the matrix representing the system of equations for each case. Another way to implement Gauss-Seidel and SOR is as recurrence relations Gauss-Seidel can be written as

$$\psi_{i,j}^{(k+1)} = \frac{\psi_{i,j+1}^{(k)} + \psi_{i,j-1}^{(k+1)} + \psi_{i+1,j}^{(k)} + \psi_{i-1,j}^{(k+1)}}{4}.$$

Successive Over Relaxation can be written as

$$\psi_{i,j}^{(k+1)} = \psi_{i,j}^{(k)} + \omega \frac{\psi_{i,j+1}^{(k)} + \psi_{i,j-1}^{(k+1)} + \psi_{i+1,j}^{(k)} + \psi_{i-1,j}^{(k+1)} - 4\psi_{i,j}^{(k)}}{4}.$$

The reason there is the symbol $(k+1)$ over some of the $\psi_{i,j}$ values and (k) over others indicates which iterate one is looking at. If one is running through the variables, then to compute the *next* ,i.e. $(k+1)$ value of $\psi_{i,j}$, one already knows the values at $\psi_{i-1,j}$ and $\psi_{i,j-1}$.

It can be shown that the optimal ω for Successive Over-Relaxation of the system obtained from discretizing Laplace's Equation is

$$\omega_{\text{opt}} = \frac{4}{2+\sqrt{4-c^2}}, \qquad \text{where } c = \cos\left(\frac{\pi}{n-1}\right) + \sin\left(\frac{\pi}{m-1}\right) \tag{12}$$

where the number of grid separations is n and m.

You should write an m-file which implements SOR and use it to solve the numerical problems with Gauss-Seidel ($\omega = 1$) and then determine the optimal ω from (12) and run SOR with this value to show the solution converges more rapidly.

Assignment

Theoretical Fluid Dynamics

1. Show that $\psi(x, y) = xy$ is an exact solution of the boundary value problem for the stream function for flow around a corner.
2. Write down a boundary value problem for the stream function describing the flow into a semi-infinite channel of width π found at $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, y > 0$.
3. Show that $\psi(x, y) = \sinh(y)\cos(x)$ is an exact solution of this boundary value problem.
4. Write down a boundary value problem for $\psi(r, \theta)$, the stream function for flow into an α-wedge.
5. Show that $\psi(r, \theta) = r^{\pi/\alpha} \sin(\frac{\pi\theta}{\alpha})$ is an exact solution of this boundary value problem.

Computational Fluid Dynamics

1. Discretize the boundary value problem for Situation A and Situation B with $n = m = 5$
2. Write down the system of equations for the discrete version of Situation A and Situation B
3. Use Gauss-Seidel Iteration (i.e. SOR with $\omega = 1$) to produce a discrete solution to Situation A and Situation B
4. Use SOR with optimal ω to produce a discrete solution to Situation A and Situation B
5. Use SOR with optimal ω and $n = m = 10$ and $n = m = 25$ to produce a discrete solution to Situation A and Situation B
6. Discuss the relationship between the error and the number of iterations SOR takes to produce a converged solution (tabulate or graph your results) for each situation

EXTRA CREDIT

1. Write down a boundary value problem for $\psi(r, \theta)$, the stream function for flow across a unit cylinder at the origin (situation D).
2. Show that the function $\psi(r, \theta) = (r - 1/r)\sin(\theta)$ solves the boundary value problem for Situation D
3. Discretize the boundary value problem for Situation C and Situation D with $n = m = 5$
4. Write down the system of equations for the discrete version of Situation C and Situation D
5. Use SOR with optimal ω to produce a discrete solution (with $n = m = 25$) to Situation C and D

Report

Write a concise report containing the following sections.

1. **Problem Overview**: A brief statement of the project objective and a summary of the steps you used to achieve it.

2. **Mathematical Formulation**: Summarize the equations used in your analysis. Describe each variable in words. Be sure to identify the role of each equation in the overall analysis.

3. **Program Listings**: You should produce m-files, like `stream.m` which have an output of:

   (i) a contour plot of $\psi(x, y)$ or $\psi(r, \theta)$

   (ii) The error between the numerical solution and the exact solution

   (iii) The discrete converged solution, in matrix form $\psi_{i,j}$

   and has an input:

   (i) n, number of horizontal divisions, m number of vertical divisions

   (ii) ω, the SOR parameter

   (iii) k, the number of SOR iterations to execute

   The purpose of each `m-file` should be stated in the text of your report. Code listings, especially those that span multiple pages, should appear in an Appendix. The input and output variables for the modules need not be described separately as long as they are adequately documented in the function prologue.

4. **Results and Discussion**: Provide answers to the questions posed in the *Assignment* section, above. Your report need not following the numbering convention in the *Assignment* so long as all the issues raised there are discussed.

5. **Feedback on Group Dynamics**: Provide a summary of how your group worked together, summarizing how many meetings occurred, how long they lasted, who was responsible for which sections of the project, et cetera. This could be done through separate paragraphs, authored by each group member.

6. **Conclusion**: In one crisp paragraph, summarize the results of this project. *Do not present new information in the Conclusion.*

The report is to be delivered in hard copy by 5:00 PM on the due date for the project.

Submission of Code

In addition to the written report, the final working version of your MATLAB programs, along with basic instructions for running them, are to be included on a disk submitted with the report. The instructions should be contained in a *plain text file* (no MS Word, no HTML formatting) with a name like `ReadMe.txt`. The instructions should briefly (one or two sentences should do) describe how to run your code. Be sure to specify any input parameters that may be needed. When I run your code(s) I should be able to recreate all the results in your report. I should not have to edit your code(s) to produce your results.

Grading Criteria

The following criteria will be used to grade the term project

Category	Points
Technical content	
Verification of theoretical results	20
Computational Fluid Dynamics	20
Numerical Results	
Implementation of SOR	10
Contour Plots	10
Tabulated Results of Running Program	20
Documentation	
Organization and documentation of m-files	5
Discussion of group dynamics	5
Grammar, style, spelling	10
Total	100
Extra Credit	
Cylindrical Problem (Situation D)	15
Numerical Solution of Situation C and D	25
Extra Credit Total	40

Report Style

The following items fall under the category of "style."

- The report should be organized into major sections.
- The text should be written in complete sentences. It should be free of slang. All abbreviations and acronyms should be defined.
- Figures must have captions. Axes must have labels. Figures and tables of results may be placed at the end of the text body, but should not be placed in an appendix. All figures and tables of results that are not discussed in the body of the text will be ignored.
- Pages in your report should be numbered.
- Only items of secondary importance are put in an appendix.

To simplify your report, assume that the reader

- is familiar with the fluid dynamics,
- is a competent MATLAB user,
- is unimpressed by fancy report covers,
- is much more interested in technical content than in font selection, three-dimensional graphics, and maximum vectorization of MATLAB code.

Do not assume that the reader has a copy of the assignment sheet. This requires, for example, that you define all variables and constants that appear in any equations you present.