Test 2: Numerical Analysis

Math 370 Friday November 8, 2002

Name: ______________________

Directions: Read *ALL* 3 (three) problems first before answering any of them. This is an open-notes, open-book test. This test has 6 pages. You must show all relevant work to support your answers. Use complete English sentences and indicate your final answer from your "scratch work." I designed it to be completed in 1 hour but you have 90 minutes.

No.	Score	Maximum
1		30
2		30
3		40
Total		100

1. *[30 points total.]* **Picard Iteration.**

Consider the Buckmire Algorithm for computing $\sqrt{R}$.

$$x_{n+1} = g(x_n) = \frac{x_n(x_n^2 + 3R)}{3x_n^2 + R}$$

(a) *[8 points].* Show that if the functional iteration scheme converges, it converges to $\sqrt{R}$. In other words, is $\lim_{n\to\infty} x_n = \sqrt{R}$?

(b) *[8 points].* Compute $g'(x)$

(c) *[6 points].* Use your knowledge of $g'(p)$, where p is the fixed point of $g(x)$, to show that Buckmire's Method is superlinearly convergent.

(e) *[8 points].* Use Buckmire's Square Root Algorithm to obtain an estimate of $\sqrt{2}$ to 6 decimal places. (Use $x_0 = 1$ as initial guess.)

3. *[30 points total.]* **Matrix Norms.**

Consider the diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$

(a) Show that $||A||_\infty = ||A||_1$ for this diagonal matrix.

(b) Show that $||A||_2 = \sqrt{\max\limits_{1 \le i \le n} \lambda_i^2}$ for the diagonal matrix A.

(c) Show that $||A||_2 = ||A||_1$ for this diagonal matrix.

4. *[40 pts. total]* **Iterative Solution of Nonlinear Systems.**

(a) *[20 points].* Show that Newton's Method applied to $\begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \vec{f}(\vec{x}) = \vec{0}$ can be written in Fixed Point Iteration form as $\vec{x} = \vec{G}(\vec{x})$ or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix}$$

where

$$\begin{aligned} g_1(x,y) &= x - \frac{f_1(x,y)\frac{\partial f_2}{\partial y}(x,y) - f_2(x,y)\frac{\partial f_1}{\partial y}(x,y)}{\det(J(x,y))} \\ g_2(x,y) &= y - \frac{f_2(x,y)\frac{\partial f_1}{\partial x}(x,y) - f_1(x,y)\frac{\partial f_2}{\partial x}(x,y)}{\det(J(x,y))} \end{aligned}$$

(b) *[10 points].* Consider the system

$$f_1(x,y) = x^2 - y - 0.2 = 0$$
$$f_2(x,y) = y^2 - x - 0.3 = 0$$

Use the formula in part (a) to write Newton's Method for this nonlinear system as a fixed point iteration scheme $\vec{x}_{k+1} = \vec{G}(\vec{x}_k)$.

(c) *[10 points].* Starting with $\vec{x}_0 = (0, 0)$ compute $\vec{x}_1$ and $\vec{x}_2$.