

Test 1: Numerical Analysis

Math 370

Wednesday October 9, 2002

Name: _____

Directions: Read *ALL* 4 (four) problems first before answering any of them. This is a one hour, open-notes, open book, test. This test has 6 pages. You must show all relevant work to support your answers. Use complete English sentences and indicate your final answer from your “scratch work.”

No.	Score	Maximum
1		20
2		20
3		20
4		40
Total		100

1. [20 points total.] Order Notation and Taylor Series.

Consider the integrally-defined function $F(h) = \int_0^h \sin(x^2) \, dx$

In this problem our goal is to describe the behavior of this given function $F(h)$ for very small values of h , i.e. $|h| \ll 1$, or as $h \rightarrow 0$.

(a) [6 points]. Write down the first three non-zero terms of the Maclaurin Series for $f(x) = \sin(x^2)$

(b) [6 points]. Use your answer in part (a) to write down the first three non-zero terms of the Maclaurin series for $F(h)$.

(c) [2 points]. Use your previous answers to show that you can write down the behavior of the integrally-defined function as $F(h) = L + \mathcal{O}(h^p)$ (Give values for L and p).

(d) [6 points]. Use L'Hôpital's Rule to confirm your answer in part (c) by using the limit definition of "big oh."

[HINT: Recall the fundamental theorem of calculus: $\frac{d}{dx} \int_a^x f(t) \, dt = \text{????}$]

2. [20 points total.] **Sequences and Limits.** Consider the sequence $x_{n+1} = \sqrt{a + x_n}$ with $x_0 = \sqrt{a}$, $a > 0$.

(a) [5 points]. Write down x_1, x_2, x_3, \dots

(b) [8 points]. Assume $x_\infty = \lim_{n \rightarrow \infty} x_n = x$. What is $\lim_{n \rightarrow \infty} x_{n+1}$? Therefore obtain an algebraic expression for x in terms of a .

(c) [5 points]. Now suppose $a = 2$. Use your calculator to compute the first 6 or so elements of the sequence. What do you estimate $\lim_{n \rightarrow \infty} x_n$ is numerically?

(d) [2 points]. Use the formula you derived in **(b)** to compare the exact value of the limit of the sequence when $a = 2$ to the approximate limit you found in **(c)**. (Are they the same or different? What do you expect?)

3. [20 points total.] **TRUE or FALSE.**

Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counter example for which the statement is not true is best. If you think the answer is **TRUE** you should also explain why you believe the statement.

(a) If $\{p_n\}$ is linearly convergent to $p = 0$ and $\{q_n\}$ is quadratically convergent to $q = 0$ then $\lim_{n \rightarrow \infty} \frac{q_n - q}{p_n - p} = 0$.

(b) The **machine precision** i.e., the number ϵ_m such that $1 + \epsilon_m = 1$, is the same on all computers.

4. [40 pts. total] **Root-Finding Methods, Convergence Criteria.**

Consider the function $J_0(x)$, the zeroth-order Bessel's Function of the First Kind. $J_0(x)$ is defined as one of the solutions to the differential equation $x^2u'' + xu' + x^2u = 0$. Two different methods, **Method A** and **Method B**, are used to approximate z_{01} the first zero of $J_0(x)$ to 7 decimal places.

Method A			
n	x_n	$f(x_n)$	$ x_n - x_{n-1} $
1	2.00000000	2.239e-001	2.000e+000
2	3.00000000	-2.601e-001	1.000e+000
3	2.50000000	-4.838e-002	5.000e-001
4	2.25000000	8.275e-002	2.500e-001
5	2.37500000	1.558e-002	1.250e-001
6	2.43750000	-1.685e-002	6.250e-002
7	2.40625000	-7.393e-004	3.125e-002
8	2.39062500	7.394e-003	1.563e-002
9	2.39843750	3.321e-003	7.813e-003
10	2.40234375	1.289e-003	3.906e-003
11	2.40429688	2.745e-004	1.953e-003
12	2.40527344	-2.325e-004	9.766e-004
13	2.40478516	2.097e-005	4.883e-004
14	2.40502930	-1.058e-004	2.441e-004
15	2.40490723	-4.240e-005	1.221e-004
16	2.40484619	-1.071e-005	6.104e-005
17	2.40481567	5.131e-006	3.052e-005
18	2.40483093	-2.790e-006	1.526e-005
19	2.40482330	1.170e-006	7.629e-006
20	2.40482712	-8.100e-007	3.815e-006
21	2.40482521	1.802e-007	1.907e-006
22	2.40482616	-3.149e-007	9.537e-007
23	2.40482569	-6.734e-008	4.768e-007
24	2.40482545	5.643e-008	2.384e-007
25	2.40482557	-5.453e-009	1.192e-007
26	2.40482551	2.549e-008	5.960e-008
27	2.40482554	1.002e-008	2.980e-008

Method B			
n	x_n	$f(x_n)$	$ x_n - x_{n-1} $
1	2.86297144	-2.102e-001	1.137e+000
2	1.58493225	4.640e-001	1.278e+000
3	2.46453524	-3.060e-002	8.796e-001
4	2.41011108	-2.741e-003	5.442e-002
5	2.40475677	3.571e-005	5.354e-003
6	2.40482563	-3.940e-008	6.887e-005
7	2.40482556	-5.636e-013	7.589e-008
8	2.40482556	0.000e+000	1.085e-012

(a) [10 pts] From the data in the two tables, what is the value of z_{01} the first root of $J_0(x)$ to 7 decimal places? How confident are you in your answer? Explain.

(b) [10 points] Which of these methods is linearly convergent? Is the other method superlinearly convergent or quadratically convergent or even faster? Can you tell? How?

Consider the following possible termination criteria when answering part (c) and (d).

Criteria 1: $|f(x_n)| \leq FTOL$

Criteria 2: $|x_n - x_{n-1}| \leq XTOL$

Criteria 3: $|f(x_n)| \leq FTOL$ AND $|x_n - x_{n-1}| \leq XTOL$

Criteria 4: $|f(x_n)| \leq FTOL$ OR $|x_n - x_{n-1}| \leq FTOL$

(c) [10 points] The data in the table is consistent with using which of the termination criteria? Note $FTOL = XTOL = 5.000\text{e-}008$ in this problem.

(d) [10 points] Suppose $FTOL = 5.000\text{e-}004$ and $XTOL = 5.000\text{e-}004$. Where would each of the Methods have terminated if each of the 4 criteria had been applied?