## *Worksheet 23*

**TITLE** Dissipative Systems
**CURRENT READING** Blanchard, 5.3 & 5.4

**Homework Assignments due Monday November 18**
**Section 5.3**: 2, 9, 12, 13, 14, 18*.
**Chapter 5 Review**: 1, 2, 6, 7, 8, 9, 11, 12, 26, 27*.
**Section 6.1**: 2, 3, 5, 7, 8, 9, 15, 18, 25*.
**Section 6.2**: 1, 2, 4, 8, 15, 16, 18*.

### SUMMARY
We shall continue our analysis of non-linear systems by introducing the concept of a Lyapunov function and learn about gradient systems.

### EXAMPLE
Recall the ODE for the damped harmonic oscillator $y'' + py' + qy = 0$ written as a system of ODEs

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -pv - qy$$

Recall that the function $H(y,v) = \frac{1}{2}v^2 + \frac{1}{2}qy^2$ is a Hamiltonian for the system when $p = 0$. However, what is $\frac{dH}{dt}$ now?

$$\frac{dH}{dt} = \frac{\partial H}{\partial y}\frac{dy}{dt} + \frac{\partial H}{\partial v}\frac{dv}{dt}$$
$$= (qy)v + v(-pv - qy)$$
$$= -pv^2$$

Which, when $p > 0$ implies that the quantity $H(y,v) = \frac{1}{2}v^2 + \frac{1}{2}qy^2$ decreases with time along solution curves of the given system. Such a function is not known as a Hamiltonian function but a Lyapunov function. Lyapunov functions are often used to make conclusions about the stability of equilibria of nonlinear systems of DEs.

### DEFINITION: Lyapunov Function
A function $L(x,y)$ is called a **Lyapunov function** for a system of differential equations, if, for every solution $(x(t), y(t))$ that is not an equilibrium solution of the system,

$$\frac{d}{dt}L(x(t), y(t)) \leq 0$$

for every $t$ with strict inequality except for a discrete set of values for $t$.

## 1. **Gradient Systems**

A system of differential equations is known as a **gradient system** if there exists a function $G(x, y)$ such that for every $(x, y)$

$$\frac{dx}{dt} = \frac{\partial G}{\partial x}$$
$$\frac{dy}{dt} = \frac{\partial G}{\partial y}$$

If $(x(t), y(t))$ are solutions of gradient system, then

$$\begin{aligned}
\frac{dG}{dt} &= \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial G}{\partial x}\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{\partial G}{\partial y} \\
&= (G_x)^2 + (G_y)^2 \\
&\geq 0
\end{aligned}$$

This should makes us realize that if we want to form a Lyapunov function for a gradient system all we need to do is select $L(x, y) = -G(x, y)$! So, **all gradient systems possess a Lyapunov function**. ( The converse is NOT true, i.e. every system with a Lyapunov function is NOT a gradient system.)

$\boxed{\textbf{EXAMPLE}}$

Let's show that $L(x, y) = -G(x, y)$ is a Lyapunov function for any gradient system
$\dot{x} = G_x, \quad \dot{y} = G_y.$

Exercise

**Blanchard, Devaney & Hall, 5.4.1, page 524.**

Consider

$$\frac{dx}{dt} = -x^3$$
$$\frac{dy}{dt} = -y^3$$

**(a)** Show that $L(x, y) = \frac{1}{2}(x^2 + y^2)$ is a Lyapunov function for the given system.

**(b)** Sketch the level sets of $L(x, y)$

**(c)** What can you conclude about the phase portrait of the system given your information from (a) and (b)? [Think about what happens to solutions as $t$ goes to infinity?]

## 2. Properties of Gradient Systems

**Gradient Systems can not possess periodic solutions!**

By using Linearization, we can show that the eigenvalues of the Jacobian of a gradient system evaluated at its equilibria will always be real (i.e. not complex) and thus solution curves of gradient systems will never be periodic.

**Not All Systems That Have Lyapunov Functions Are Gradient Systems**

$\boxed{\textbf{EXAMPLE}}$

The following system has a Lyapunov function of $L(x,y) = x^2 + y^2$ but is NOT a gradient system.

$$
\begin{aligned}
\frac{dx}{dt} &= -x + y \\
\frac{dy}{dt} &= -x - y
\end{aligned}
$$

Let's Prove This Result.