Differential Equations

Math 341 Fall 2013 MWF 12:50-1:45pm Fowler 307

http://faculty.oxy.edu/ron/math/341/13/

Worksheet 18

TITLE Linear Systems with Complex Eigenvalues
CURRENT READING Blanchard, 3.4

Homework Assignments due Monday November 4 (* indicates EXTRA CREDIT)
Section 3.3: 3, 4, 7, 8, 20*.
Section 3.4: 1, 2, 3, 4, 16*, 23*.
Section 3.5: 3, 4, 9, 10, 12, 18*, 23*.

SUMMARY
We'll continue to explore the various scenarios that occur with linear systems of ODEs. This time dealing with those that possess complex (or imaginary) eigenvalues.

1. Two Complex Eigenvalues

Given a system of linear ODEs with associated matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the characteristic polynomial is $p(\lambda) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \text{tr}(\text{A})\lambda + \det(\text{A}) = A\lambda^2 + B\lambda + C = 0$ where $A = 1$, $B = -(a+d)$ and $C = ad - bc$.

Using the quadratic formula to solve the characteristic quadratic polynomial equation $A\lambda^2 + B\lambda + C = 0$ produces $\lambda = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

Previously we showed that the conditions on A, B and C for two real eigenvalues was $B^2 > 4AC$ which is equivalent to $(a-d)^2 > -4bc$.

Clearly, the condition $B^2 < 4AC$ which is equivalent to $(a-d)^2 < -4bc$ will result in "imaginary" solutions to the characteristic polynomial, i.e. complex eigenvalues.

Also (for completeness), when $(a-d)^2 = -4bc$ there will be only one solution to the quadratic equation, i.e. repeated eigenvalues equal to $\lambda = \dfrac{(a+d)}{2}$.

2. Reviewing Complex Arithmetic

Recall that the $i^2 = -1 \Leftrightarrow i = \sqrt{-1}$ and a complex number z has the form $z = a + bi$ where a and b are real numbers. We refer to the real part of z as $\text{Re}z$ and is equal to a and the complex part of z as $\text{Im}z$ and is equal to b.

De Moivre's Formula

One of the most fun formulas in mathematics is DeMoivre's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

When $\theta = \pi$ one gets $e^{i\pi} = -1$ which means that $e^{i\pi} + 1 = 0$ which has all of these important numbers in it: 0, 1, π, i and e. Some call it the greatest equation, ever!

Complex Conjugates

Complex numbers often show up in pairs, called complex conjugates. A complex conjugate of the number $z = a+bi$ is $z^* = a-bi$. For example, $\sqrt{-4} = 0 \pm 2i$. These numbers $-2i$ and $2i$ are complex conjugates of each other. Note that the product of two complex conjugates is a completely real number.

GROUPWORK

Let's practice some complex arithmetic by simplifliying some expressions into the form $a+bi$.

1. $(1+2i)(1-2i)$

2. $\frac{4}{i}$

3. $e^{-i\pi/2}$

4. $(-1+2i)(2-3i)$

RECALL

The general solution to $\frac{d\vec{x}}{dt} = A\vec{x}$ is $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$ where $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$, i.e $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to eigenvalues λ_1 and λ_2.

What happens if the eigenvalues are complex? Let's find out!

EXAMPLE

What's the general solution to $\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \vec{x}$. Check your answer.

3. Types Of Solutions With Complex Eigenvalues

Given that one has complex eigenvalues of the form $\alpha \pm i\beta$ there are three possibilities for what the phase portrait behavior of the solution will look like. All the solutions will rotate around the origins with a period of $\frac{2\pi}{\beta}$.

CASE 1: $\alpha < 0$ All solutions spiral inwards towards the origin, which is classified as a **stable spiral sink**.

CASE 2: $\alpha > 0$ All solutions spiral outwards from the origin, which is classified as an **unstable spiral source**.

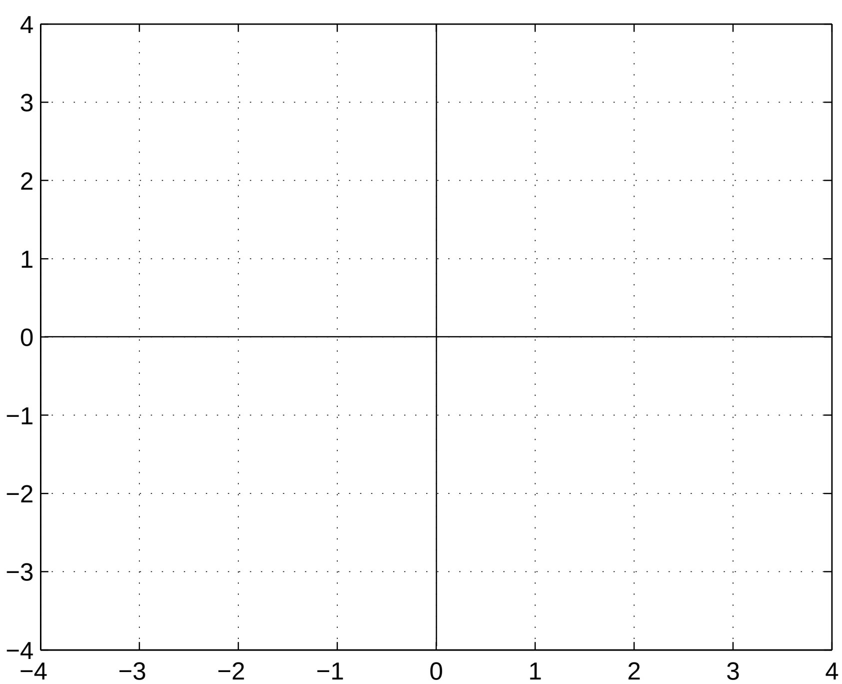
CASE 3: $\alpha = 0$ All solutions follow elliptical or circular paths rotating around the origin, which is classified as an **unstable center**.

GROUPWORK

Come up with your own examples of 2x2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ that have complex eigenvalues and the corresponding ODE for each of the three cases named above. Use `HPGSystemsSolver` or `pplane` to help you sketch the phase portrait for each case on the given axes.

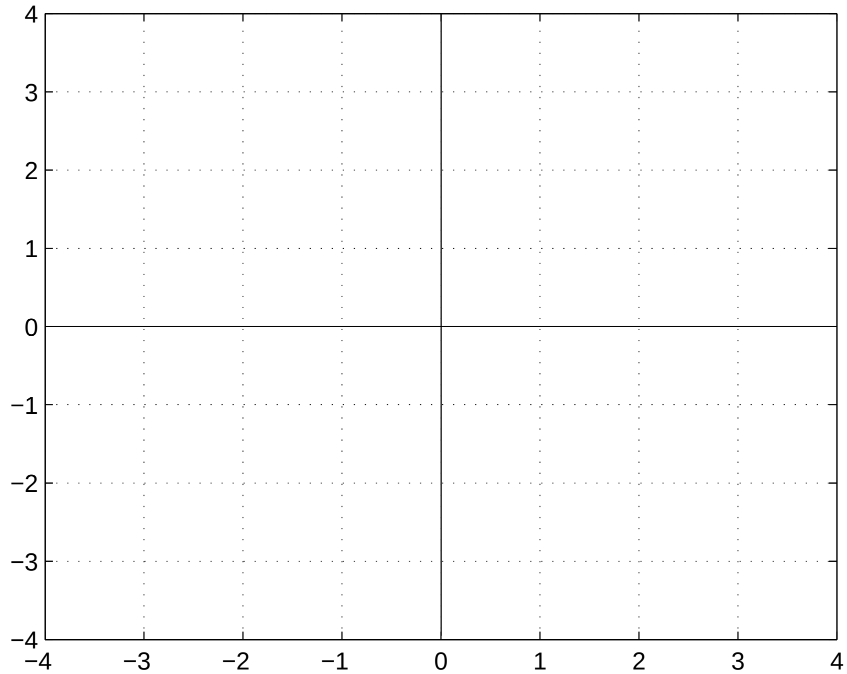
Recall, $\lambda = \alpha \pm \beta i = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$. What values of a,b,c and d will produce α equalling zero? α negative? α positive? Or you could just explore with parameters until you obtain the phase portrait you are looking for in each case.

4. CASE 1: spiral sink



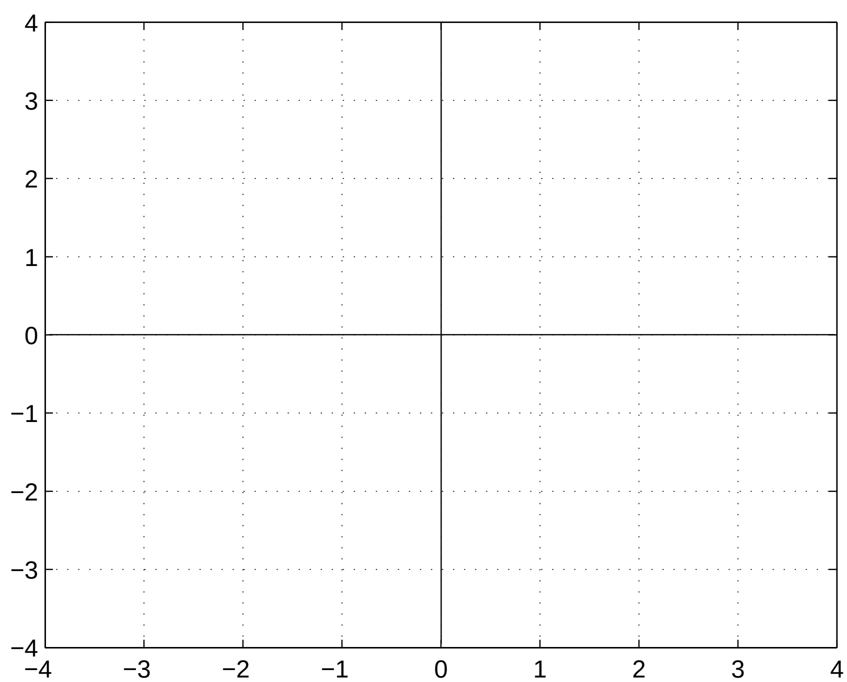
Corresponding ODE: $\frac{d\vec{x}}{dt} = \begin{bmatrix} \quad & \quad \end{bmatrix} \vec{x}$

5. **CASE 2: spiral source**



Corresponding ODE: $\frac{d\vec{x}}{dt} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \vec{x}$

6. **CASE 3: unstable center**



Corresponding ODE: $\frac{d\vec{x}}{dt} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \vec{x}$

Given that the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_1 = 1 - 2i$. Which of the following could be the set of eigenvectors associated with A?

(a) $\{\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} -1 \\ i \end{bmatrix}\}$

(b) $\{\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}\}$

(c) $\{\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}\}$

(d) All of the above.

(e) None of the above.

The matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_1 = 1 - 2i$ with associated eigenvectors $\begin{bmatrix} i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. A solution to $\frac{d\vec{Y}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \vec{Y}$ is given by

(a) $Y = \begin{bmatrix} e^t \cos(2t) \\ e^t \sin(2t) \end{bmatrix}$

(b) $Y = \begin{bmatrix} e^{2t} \cos(t) \\ e^{2t} \sin(t) \end{bmatrix}$

(c) $Y = \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix}$

(d) $Y = \begin{bmatrix} e^t \cos(t) \\ e^t \sin(t) \end{bmatrix}$

(e) None of the above.