Test 2: Differential Equations

Math 341 Fall 2009 | Friday November 20

 2:30pm-3:25pm

Name: BUCKMIRE

Directions: Read *all* problems first before answering any of them. There are 6 pages in this test. This is a 55-minute, no-notes, closed book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

Offer: If there is a formula or piece of information that you feel that you need in order to solve a problem, I will provide it to you at a non-negotiable rate of at least a one point deduction.

Pledge: I, ______________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		15
2		15
3		20
BONUS		5
Total		**50**

1. *[15 points total.]* **Linear Systems of Differential Equations, Trace-Determinant Plane, Bifurcation.** VISUAL & ANALYTIC.

Consider $\dfrac{d\vec{x}}{dt} = A\vec{x}$ where $A = \begin{bmatrix} \alpha & -1 \\ 1 & \alpha \end{bmatrix}$ and α is a known real-valued parameter. Recall that the eigenvalues of the matrix A are given by $\lambda = \dfrac{-T \pm \sqrt{T^2 - 4D}}{2}$ where T is the trace of A and D is the determinant of A.

1(a) *[5 points].* Show that for all α values in this matrix A, $T^2 - 4D + 4 = 0$.

$T = 2\alpha$

$D = \alpha^2 + 1$

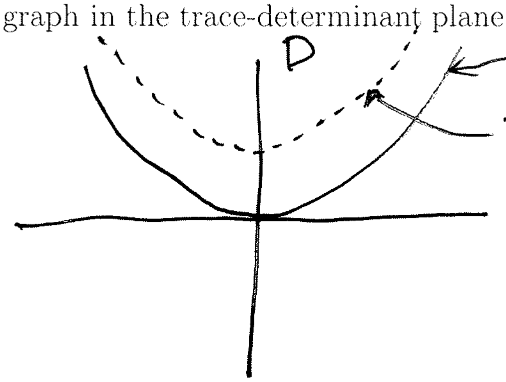
$\text{LHS} = T^2 - 4D + 4 = (2\alpha)^2 - 4(\alpha^2+1) + 4$

$= 4\alpha^2 - 4\alpha^2 - 4 + 4 = 0 = \text{RHS}$

1(b) *[5 points].* Sketch a graph in the trace-determinant plane carved out by varying values of α.

$\lambda = \dfrac{-2\alpha \pm \sqrt{-4}}{2}$

$\lambda = -\alpha \pm i$

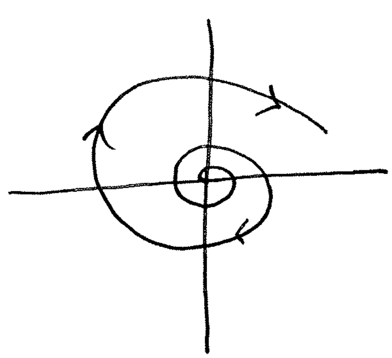


i.e. $D = \dfrac{T^2}{4} + 1$

1(c) *[5 points].* Does the qualitative nature of the phase portrait (and equilibrium at the origin) change as α varies? If so, **give all the bifurcation values of** α, **classify the equilibrium point** for values of α (greater than, less than and equal to the bifurcation value(s)) and **provide reasonable sketches of the phase portrait(s))** in each case in the space below.

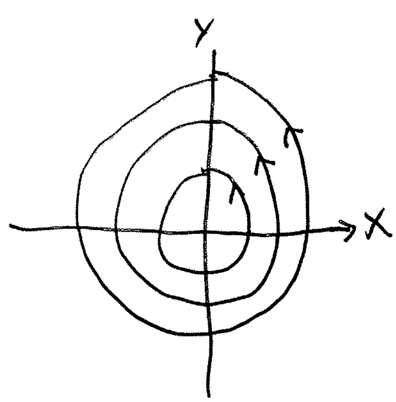
Nature of phase portrait changes as α goes from negative to positive, i.e $\alpha_B = 0$

$\alpha < 0$



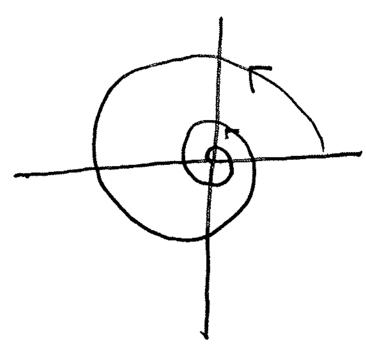
SPIRAL SOURCE

$\alpha = 0$



CENTER

$\alpha > 0$



SPIRAL SINK

2. *[15 points total.]* **Linearization, .** ANALYTIC, VERBAL & VISUAL.
Are the following statements TRUE or FALSE – put your answer in the box. To receive ANY credit, you must also give a brief, and correct, explanation in support of your answer! For example, if you think the answer is **FALSE** providing a counterexample for which the statement is NOT TRUE is best. If you think the answer is **TRUE** you should prove why you think the statement is always true. Your explanation of your answer is worth FOUR TIMES as much as the answer you put in the box.

Consider the system

$$\frac{dx}{dt} = \alpha x - y = f(x,y)$$
$$\frac{dy}{dt} = x + \alpha y + x^2 = g(x,y)$$

where α is a known real-valued parameter.

2(a) TRUE or FALSE? "A Hamiltonian function $H(x, y)$ for the given nonlinear system of ODEs exists."

$\boxed{\text{TRUE}}$

$f = H_y$ and $g = -H_x$

$f_x = -g_y$

$f_x = \alpha$

$g_y = \alpha \Rightarrow -g_y = -\alpha$

$\alpha = -\alpha$ only when $\alpha = 0$.

2(b) TRUE or FALSE? "A gradient function $G(x, y)$ for the nonlinear system of ODEs exists."

$\boxed{\text{FALSE}}$

$f = G_x \quad g = G_y$

$f_y = g_x$

$f_y = -1$

$g_x = 1 + 2x$

$-1 \neq 1 + 2x$

No Gradient Function!

2(c) TRUE or FALSE? "The origin of the phase portrait for this nonlinear system will look like a center (a series of concentric circles)."

$\boxed{\text{FALSE}}$

$$J = \begin{pmatrix} \alpha & -1 \\ 1+2x & \alpha \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}$$

When $\alpha = 0$ the origin will be a center (circles)
when $\alpha \neq 0$, the origin will be spirals

3. *[20 points total.]* **Linear Systems of Differential Equations, Matrix Exponential, General Solution.** ANALYTIC & VERBAL.

Consider the initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ where $\vec{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\vec{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

The solution can be written as $\vec{x}(t) = e^{At}\vec{x}_0$, which we will call the **matrix exponential solution**.

The goal of this problem is to show that the general solution to the given initial value problem (which we will call the **general eigenvector solution**) can be represented using the matrix exponential e^{At}.

Recall from Calculus that $\frac{d}{dt}e^{\square t} = \square e^{\square t}$ as long as $\frac{d}{dt}\square = 0$.

Recall from Linear Algebra that if A is diagonalizable, then the matrix exponential $e^{At} = Se^{\Lambda t}S^{-1}$ where S is an $n \times n$ matrix whose columns consist of the n eigenvectors $\vec{v}_1, \ldots \vec{v}_n$ of A, and Λ is an $n \times n$ diagonal matrix with the corresponding eigenvalues along the diagonal.

$$e^{At} = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \ldots & \vec{v_n} \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_3} & \ldots & \vec{v_n} \end{bmatrix}^{-1}$$

Suppose that A is a 2×2 diagonalizable matrix of the form $\begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$ and A has two eigenvalues λ_1 and λ_2 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$.

3(a) *[5 points].* Show that the matrix exponential solution $\vec{x}(t) = e^{At}\vec{x}_0$ satisfies the given initial value problem.

Satisfies the DE $\quad$ LHS: $\frac{d\vec{x}}{dt} = \frac{d}{dt}(e^{At}\vec{x}_0) = Ae^{At}\vec{x}_0$

$= A\vec{x} = $ RHS

LHS = RHS ✓

I.C. $\quad$ RHS $= \vec{x}(0) = e^{A\cdot 0}\vec{x}_0$

$= I\vec{x}_0$

$= \vec{x}_0 = $ RHS

LHS = RHS ✓

3(b) *[5 points]*. Show that in order for the general eigenvector solution to satisfy the given initial value problem, the linear system $\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ must be solved to find the unknown constants c_1 and c_2.

Satisfy the IC

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

$$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Satisfy the DE

$$\frac{d\vec{x}}{dt} = c_1 \lambda_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 \lambda_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

$$= c_1 e^{\lambda_1 t} A \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} A \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} = A\left[c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}\right]$$

$$= A\vec{x} \checkmark$$

$$= RHS$$

3(c) *[10 points]*. Use your results from (a) and (b) to show that the matrix exponential solution is identical to the general eigenvector solution for the given initial value problem $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = \vec{x_0}$ where $A = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$ is a diagonalizable matrix.
EXPLAIN YOUR ANSWER and SHOW ALL YOUR WORK.

$$\vec{x} = e^{At}\vec{x}_0 \leftarrow \text{matrix exponential sol}^{n}$$

$$= \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} \quad (\text{from 3b})$$

$$= \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \leftarrow \text{general eigenvector solution}$$

By Existence & Uniqueness Theorem we know there is a unique solution so these "two" sol^{n}s must be identical.

BONUS. Find the matrix exponential solution to the initial value problem $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and simplify it into the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

HINT: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\lambda^2 - T\lambda + D = 0$

$\lambda^2 - (-3)\lambda + 2 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0$

$\lambda = -1, -2$

$E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$

$E_{-2} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$

$\vec{x} = \cancel{\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}}$

$$\vec{x} = e^{At}\vec{x}_0$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\frac{1}{-2+1}\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 3e^{-t} \\ -2e^{-2t} \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{pmatrix}$$

Check: $\dfrac{d\vec{x}}{dt} = \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ 3e^{-t} + 8e^{-2t} \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{pmatrix}$

$$= \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ -6e^{-t} + 4e^{-2t} + 9e^{-t} - 12e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -3e^{-t} + 4e^{-2t} \\ 3e^{-t} - 8e^{-2t} \end{pmatrix}$$