Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 2

 http://faculty.oxy.edu/ron/math/341

Class 30: Monday April 18

TITLE *The Method of Frobenius*
CURRENT READING Zill 6.2

Homework Set #12
Zill, Section 6.1: 10*,13*,15*,26*,29*
Zill, Section 6.2: 3*,4*,12*,13* *EXTRA CREDIT 33*
Zill, Section 6.3: 5*,10*,25*,30* *EXTRA CREDIT 33*
Zill, Chapter 6 Review: 4*,5*,10*,15*,20* *EXTRA CREDIT 22*

SUMMARY
The Method of Frobenius is used to find series solutions around differential equations which have regular singular points.

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1. The Method of Frobenius

THEOREM: **Frobenius' Theorem**
If a point x_0 is a **regular singular point** of the DE $y'' + P(x)y' + Q(x)y = 0$ then there exists at least one solution of the form $y = (x - x_0)^r \sum_{k=0}^{\infty} (x - x_0)^k$ where r is the root of the indicial equation of the DE.

DEFINITION: **indicial equation** The **indicial equation** is a quadratic equation which is obtained from the DE after substituting $y = \sum_{k=0}^{\infty} (x - x_0)^{k+r}$ into the DE and setting the coefficient of the lowest power of x to zero. For example, in a DE of the form $y'' + P(x)y' + Q(x)y = 0$ which has a regular singular point at $x = 0$ then the power series expansion $p(x) = xP(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ and $q(x) = x^2 Q(x) = b_0 + b_1 x + b_2 x^2 + \ldots$ lead to the inidicial equation $r(r-1) + a_0 r + b_0 = 0$. The roots r_1 and r_2 of this equation are called the **exponents at the singularity**.

EXAMPLE **Zill, page 252, Example 2**. Solve $3xy'' + y' - y = 0$ by the Method of Frobenius.

RECALL One can obtain the values a_0 and b_0 from the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)Q(x)$ by simply taking the limit as $x \to x_0$ so that $a_0 = \lim_{x \to x_0} p(x)$ and $b_0 = \lim_{x \to x_0} q(x)$. Then the indicial equation is $r(r-1) + a_0 r + b_0 = 0$.

Exercise Find the indicial equation of $2xy'' + (1+x)y' + y = 0$

EXAMPLE **Zill, page 252, Example 4**. Show that $xy'' + y = 0$ has the solution

$$y(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} x^{k+1} = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \dots$$

Case I: Two real distinct roots where $r_1 - r_2$ is NOT an integer

The solution has the form $y_1(x) = \sum_{k=0}^{\infty} c_n x^{k+r_1}$ and

$y_2(x) = \sum_{k=0}^{\infty} b_n x^{k+r_2}$ where c_0 and b_0 are non-zero.

Case II: Two real distinct roots where $r_1 - r_2$ IS an integer

The solution has the form $y_1(x) = \sum_{k=0}^{\infty} c_n x^{k+r_1}$, $c_0 \neq 0$ and

$y_2(x) = Cy_1(x)\ln(x) + \sum_{k=0}^{\infty} b_n x^{k+r_2}$, $b_0 \neq 0$ where C is a constant which may be zero in some instances.

Case III: One real root r_1

The solution has the form $y_1(x) = \sum_{k=0}^{\infty} c_n x^{k+r_1}$, $c_0 \neq 0$ and

$y_2(x) = y_1(x)\ln(x) + \sum_{k=1}^{\infty} b_n x^{k+r_1}$, $b_0 = 0$

2. Bessel's Equation

The equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, which is known as **Bessel's Equation**.

Consider the cases $\nu = 0$, $\nu = \frac{1}{2}$ and $\nu = 1$. Obtain the indicial equation for these values of ν and state which Case the roots fall under. We'll continue our exploration of this very rich equation next time.