Differential Equations

Math 341 Spring 2005
MWF 8:30 - 9:25am Fowler North 2

http://faculty.oxy.edu/ron/math/341

Class 24: Wednesday March 30

TITLE *The Laplace Transform*
CURRENT READING Zill 7.1

Homework Set #10
Zill, Section 7.1: 4*, 9*, 11*, 20* *EXTRA CREDIT 41*
Zill, Section 7.2: 2*, 7*, 11*, 25*, 41* *EXTRA CREDIT 42*

SUMMARY
We introduce a new kind of operator, an integral operator, called the Laplace Transform, which can be used to solve differential equations.

1. Introducing The Laplace Transform

DEFINITION: **Integral Transform**

If a function $f(t)$ is defined on $[0, \infty)$ then we can define an integral transform to be the improper integral $F(s) = \int_0^\infty K(s,t)f(t)\,dt$. If the improper integral converges then we say that $F(s)$ is the integral transform of $f(t)$. The function $K(s,t)$ is called the **kernel** of the transform. When $K(s,t) = e^{-st}$ the transform is called **the Laplace Transform.**

DEFINITION: **Laplace Transform**

Let $f(t)$ be a function defined on $t \geq 0$. The Laplace Transform of $f(t)$ is defined as

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

(Note the use of capital letters for the transformed function and the lower-case letter for the input function.)

EXAMPLE Let's show that $\mathcal{L}[1] = \frac{1}{s}, s > 0$

Exercise

Compute $\mathcal{L}[t]$.

2. Properties of The Laplace Transform

$\mathcal{L}$ is a linear operator, in other words $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$

EXAMPLE Let's show that $\mathcal{L}[\sin(kt)] = \dfrac{k}{s^2+k^2}$, $s > 0$

Important Laplace Transforms	
$f(t)$	$F(s) = \mathcal{L}[f(t)]$
1	$\dfrac{1}{s}$
t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$
$\sinh(kt)$	$\dfrac{k}{s^2-k^2}$
$\cosh(kt)$	$\dfrac{s}{s^2-k^2}$

Q: Does every function have a Laplace Transform? **A:** Hell, no! (i.e. t^{-1}, e^{t^2} etc

DEFINITION: **exponential order**

A function f is said to be of **exponential order** c if there exist constants c, $M > 0$, $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$.

Basically this is saying that in order for $f(t)$ to have a Laplace Transform then in a race between $|f(t)|$ and e^{ct} as $t \to \infty$ then e^{ct} must approach its limit first, i.e. $\lim_{t\to\infty} \dfrac{f(t)}{e^{ct}} = 0$.

THEOREM

If f is piecewise continuous on $[0, \infty)$ and of exponential order c, then $F(s) = \mathcal{L}[f(t)]$ exists for $s > c$ and $\lim_{s\to\infty} F(s) = 0$

Exercise Find the Laplace Transform of the piecewise function $f(t) = \begin{cases} 0, & 0 \leq t < 3 \\ 2, & t \geq 3 \end{cases}$