Differential Equations

Class 18: Wednesday March 2

TITLE *Cauchy-Euler Equations*
CURRENT READING Zill, 4.7, 4.9, Chapter 4 in Review

SUMMARY
We will conclude our study of Higher Order linear differential equations by analyzing a particular class of problems known as **Euler-Cauchy Equations**.

1. Solving Cauchy-Euler Equations

DEFINITION: **Cauchy-Euler Equation**

A n^{th} order linear differential equation which has the form
$a_n x^n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots a_2 x^2 y'' + a_1 x y' + a_0 y = g(x)$
is called a **Cauchy-Euler equation** or **equidimensional equation**.

Similar to how we analyzed constant coefficient linear DEs by looking at the cases which show up with 2^{nd} order DEs, we will focus our study of Cauchy-Euler equations by looking at $ax^2y'' + bxy' + cy = 0$.

This time, if we make the *ansatz* that the solution looks like $y = x^m$ the analysis is eerily similar to the constant coefficient case. This time, the auxiliary equation becomes $am^2 + (b - a)m + c = 0$

Again, there are three distinct types of solutions to this equation, depending on the values of a, b and c.
Case I: Two distinct real roots (m_1 and m_2)
The fundamental set of solutions will look like $y = c_1 x^{m_1} + c_2 x^{m_2}$

Case II: Two indistinct real roots (m_1 repeated)
The fundamental set of solutions will look like $y = c_1 x^{m_1} + c_2 x^{m_1} \ln(x)$

Case III: Two distinct complex roots ($m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$)
The fundamental set of solutions will look like $y = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$

EXAMPLE Solve $4x^2y'' + 17y = 0, \quad y(1) = -1, \quad y'(1) = -\frac{1}{2}$

Exercise Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$ What method will you use?

2. Transformation to Constant Coefficient DE

By using the change of variable $x = e^t$ (i.e. $t = \ln(x)$) we can show that a Cauchy-Euler equation can be converted into a constant coefficient linear DE.

EXAMPLE We can show that $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$ and $\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

Thus we can re-write $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$ as ______________________________.

Exercise Solve $x^2y'' - xy' + y = \ln(x)$

3. Some Nonlinear Equations

The most general form of the second order nonlinear DE would be $F(x, y, y', y'') = 0$. In Section 4.9 Zill points out the usefulness of the transformation $u = y'$ when one is faced with 2^{nd} order nonlinear DEs of the form $F(x, y, y'') = 0$ (noy' term) or $F(y, y', y'') = 0$ (no x term).

EXAMPLE Solve the equations $xy'' = y'$ and $yy'' = (y')^2$ by using the transformation $u = y'$.

Taylor Series Approximation Technique for Nonlinear IVPs

Consider the nonlinear DE with initial conditions: $y'' = x + y - y^2, \quad y(0) = -1, \quad y'(0) = 1$

We can obtain a 2^{nd} Degree Taylor Approximation for the exact solution $y(x)$ using the information found in the problem.

RECALL A Taylor Series for a function $y(x)$ about a point $x = a$ is given by $y(x) \approx y(a) + y'(a)(x-a) + y''(a)\dfrac{(x-a)^2}{2!} + y'''(a)\dfrac{(x-a)^3}{3!} + \ldots$

Exercise Write down the 2^{nd} Degree Taylor Approximation for $y(x)$ about the point $a = 0$, where you know that $y(x)$ solves the given initial value problem $y'' = x + y - y^2, \quad y(0) = -1, \quad y'(0) = 1$. **How would you write down the 5^{th} degree approximation?**