Differential Equations

Math 341 Spring 2005 | MWF 8:30 - 9:25am Fowler North 2

 | http://faculty.oxy.edu/ron/math/341

Class 10: Wednesday February 9

TITLE *Introduction to Bifurcation*
CURRENT READING Zill, 3.2

Homework Set #4
Zill, Section 2.4: 5, 10*, 11*, 19, 27*, 30, 32, 38*, *EXTRA CREDIT 42, 44*
Zill, Section 3.1: 1*, 10, 17*, 19*, *EXTRA CREDIT 37*
Zill, Section 3.2: 5, 13*, 19*, *EXTRA CREDIT 22*

SUMMARY
We will be introduced to another technique for finding the solution of a DE. This time we restrict the class of DEs we are trying to solve to be first-order **linear** ordinary differential equations.

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1. Parameter Sensitivity

EXAMPLE **Zill, page 109, Question 5.** Consider a model for logistic growth of fish with constant harvesting given by the IVP $P' = P(\alpha - \beta P) - h, \quad P(0) = P_0$ where α, β and h are all ≥ 0. Let $\alpha = 5$ and $\beta = 1$. Let's investigate how or if the solution changes as the values of the parameter h changes.

GroupWork Draw phase lines for the critical points of the IVP when the value of $h = 0$, 2, 4, 6 and 8. Identify and classify any and all critical points for each value of h

Is there a particular value of h for which the nature of the solution changes? If so, find it.

DEFINITION: **bifurcation**

A change in the qualitative nature of the phase portrait or long-term behavior of the solution of a differential equation when the value of a parameter changes is called a **bifurcation** of the DE. The value at which such changes occur is known as a **bifurcation point** or **bifurcation value** of the DE.

DEFINITION: **hyperbolic and nonhyperbolic critical points**

A critical point of an autonomous DE $y' = f(y)$ is said to be **nonhyperbolic** if arbitrarily small changes (known as **perturbations**) in $f(y)$ cause a bifurcation in the DE, i.e. critical points appear or disappear, or change the nature of their stability. If perturbations to $f(y)$ cause changes in the quantitative but not qualitative nature of the critical points, these critical points are called **hyperbolic**.

2. Classifying Critical Points

Consider the autonomous DE $y' = f(y)$ where $f(y)$ is a continuously differentiable function and y_0 is a critical point, i.e. $f(y_0) = 0$.
If $f'(y_0) < 0$, then y_0 is a stable critical point of the DE, also known as a **sink**.
If $f'(y_0) > 0$, then y_0 is an unstable critical point of the DE, also known as a **source**.
If $f'(y_0) = 0$, then y_0 is a semi-stable critical point also known as a **node**.

EXAMPLE Consider the one-parameter family of autonomous DE $\frac{dy}{dt} = y^3 - ay = y(y^2 - a)$, where a is a parameter which can take on any real value. Let's sketch the **bifurcation diagram** of this DE.

THEOREM

Consider a one-parameter family of autonomous DEs where $y' = f(y; \alpha)$ and α is a parameter. The value α_0 will be a **bifurcation value** if and only if $f(y; \alpha_0) = 0$ has a solution $y = y_0$ and $f'(y_0; \alpha_0) = 0$.

Exercise Draw the bifurcation diagram for the one-parameter family of autonomous DE $\frac{dy}{dt} = y(1 - y^2) + \alpha$. Identify and classify and and all critical points for all values of α.