Differential Equations

Math 341 Spring 2005 MWF 8:30 - 9:25am Fowler North 4
 http://faculty.oxy.edu/ron/math/341

Class 1: **Wednesday January 19**

TITLE Definitions and Terminology
CURRENT READING Zill, 1.1 and 1.2

Homework Assignments for Chapter 1
Section 1.1 # 1, 2, 3, 4, 8, 9, 11, 12, 15, 17, 21, 22, 27, 28, 29, 30, 31, *EXTRA CREDIT: 41, 42, 43, 44, 51* (**Hand in # 2, 4, 8, 12, 17, 22, 28, 30 ON FRI JAN 21**)
Section 1.2 # 1, 2, 3, 4, 5, 6, 15, 17, 20, *EXTRA CREDIT: 31, 32* (**Hand in # 2, 4, 6, 20 ON FRI JAN 28**)
Section 1.3 # 1, 2, 3, 5, 8, *EXTRA CREDIT: 13, 14* (**Hand in # 3, 5, 8 ON FRI JAN 28**)
CHAPTER 1 REVIEW: **Hand in # 5, 7, 8, 9, 10, 15, 16, 17, 21, 22 on FRI JAN 28**

SUMMARY
In today's class we shall go through various basic definitions and terminology associated with the study of differential equations in order to introduce you to the language we will be using throughout the course this semester.

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1. Definitions and Terminology

RECALL A function $y = f(x)$ is said to relate the dependent variable y and the independent variable x.

DEFINITION: **differential equation**
An equation containing the derivative of one or more dependent variables, with respect to one or more independent variables is said to be a **differential equation**, or DE. (Zill, Definition 1.1, page 2).

Differential Equations are classified by type, **order** and **linearity**.

TYPE
There are two main *types* of differential equation: "ordinary" and "partial". An **ordinary differential equation** (ODE) contains derivatives with respect to only one independent variable (though there may be multiple dependent variables). A **partial differential equation** contains partial derivatives with respect to multiple independent variables.

EXAMPLE
Consider the following differential equations: Classify them as either ODEs or PDEs.

$$(A)\frac{d^2u}{dx^2} + \lambda e^u = 0 \qquad (B)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (C)\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$$

RECALL: There are different notations to denote differentiation of a dependent variable y with respect to an independent variable x. **Leibniz Notation** has the form $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc. **Prime Notation** has the form $y', y'', y''', y^{(iv)}$ etc. **Dot Notation** is used (primarily by physicists and engineers) when the independent variable is t and has the form $\dot{y}, \ddot{y}$ etc. **Partial Differentiation** has the form u_x, u_{xy}, u_{xx} etc.

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ORDER

The *order* of a differential equation is the order of the highest derivative found in the DE.

$\boxed{\text{EXAMPLE}}$

What is the order of the following differential equations?

$$\frac{d^2u}{dx^2} + \left(\frac{du}{dx}\right)^3 + 4u\sin(x) = 0$$

$$\frac{du}{dx} - \left(\frac{u^2}{x^2+1}\right)^2 + \ln(x) = 0$$

$$y''' - 2e^x y'' + 5\cos(x)y' = 20$$

The most general form of an n-th order ordinary differential equation is $F(x, y, y', y'', y''', \ldots, y^{(n)}) = 0$ where F is a real-valued function of $n+2$ variables x, $y(x)$, $y'(x)$,...,$y^{(n)}(x)$.

The **normal form** of an n-th order differential equation involves solving for the highest derivative and placing all the other terms on the other side of the equation, i.e.

$$\frac{d^ny}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$

For example, during the class we shall grow very familiar with the normal form of first order ordinary differential equations, which look like: $y' = f(x, y)$.

Q: Write down the normal form of a second order ordinary differential equation here:
A:

LINEARITY

An n-th order differential equation is said to be **linear** if the function F is linear in the variables $y, y', y'', \ldots, y^{(n)}$. (HINT: what variable is missing from this list?) To be specific, an n-th order ODE will have the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

A **nonlinear** ODE is one that is not linear, i.e. does not have the above form.

$\boxed{\text{GROUPWORK}}$

Write down an example of one linear ODE and one nonlinear ODE.

2. Solutions of ODEs

DEFINITION: **solution**

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n-th order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval. (Zill, Definition 1.2, page 5).

In other words, a solution ϕ of an n-th order ODE is a function which possesses at least n derivatives and for which

$$F(x, \phi(x), \phi'(x), \phi''(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I$$

We say that ϕ *satisfies* the differential equation on I.

The interval I in the above definition is also known as the **interval of definition**, **interval of existence**, **interval of validity** or the **domain of the solution** of the ordinary differential equation $F(x, y, y', y'', \ldots, y^{(n)}) = 0$.

EXAMPLE

Consider the ODE $xy' + y = 0$. Confirm that $y = 1/x$ is the solution of the ODE. What is the domain of definition of the function $y = 1/x$? What is the interval of definition of the solution of the ODE? Are these two sets identical?

DEFINITION: **solution curve**

A graph of the solution ϕ of an ODE is called a **solution curve**. As noted in the previous example, there may be a difference between the domain of definition of the function ϕ and the interval of definition of the solution ϕ. Thus there may be a difference between the graph of the function $\phi(x)$ and the solution ϕ on the interval of definition of the ODE.

DEFINITION: **implicit solution**

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ODE on an interval I, provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on I. (Zill, Definition 1.3, page 6).

EXAMPLE

Consider the ODE $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Confirm that $x^2 + y^2 = 25$ is the implicit solution of the ODE on the open interval $(-5, 5)$. Show also that there are two explicit solutions to the ODE. Are the solution curves different from the graph of the implicit solution?

DEFINITION: **families of solutions**

A solution containing an arbitrary constant (parameter) represents a set $G(x, y, c) = 0$ of solutions to an ODE called a **one-parameter family of solutions**. A solution to an $n-th$ order ODE is a **n-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. Since the parameter can be assigned an infinite number of values, an ODE can have an infinite number of solutions.

DEFINITION: **particular solution**

A **particular solution** to an ODE is a solution which is free of arbitrary constants. In other words, a particular solution corresponds to a particular value of the parameter c.

In the example above, $G(x, y, c) = x^2 + y^2 - c = 0$ is a one-parameter family of solutions to the given ODE $y' = -y/x$. There are two explicit particular solutions for each value of the parameter c defined on the interval $(-\sqrt{c}, \sqrt{c})$.

DEFINITION: **singular solution**

A **singular solution** to an ODE is an "extra solution" which can not be obtained by specifying any of the unknown parameters in the family of solutions to the ODE.

For example, the one-parameter family of solutions to the ODE $y' = x\sqrt{y}$ on $(-\infty, \infty)$ is $y = (\frac{1}{4}x^2 + c)^2$. **Verify this!**. However, notice that the **trivial solution** $y = 0$ is also a solution to the ODE, but is not a member of the family of solutions (no value of c corresponds to $y = 0$). Thus $y = 0$ is a singular solution of the ODE.